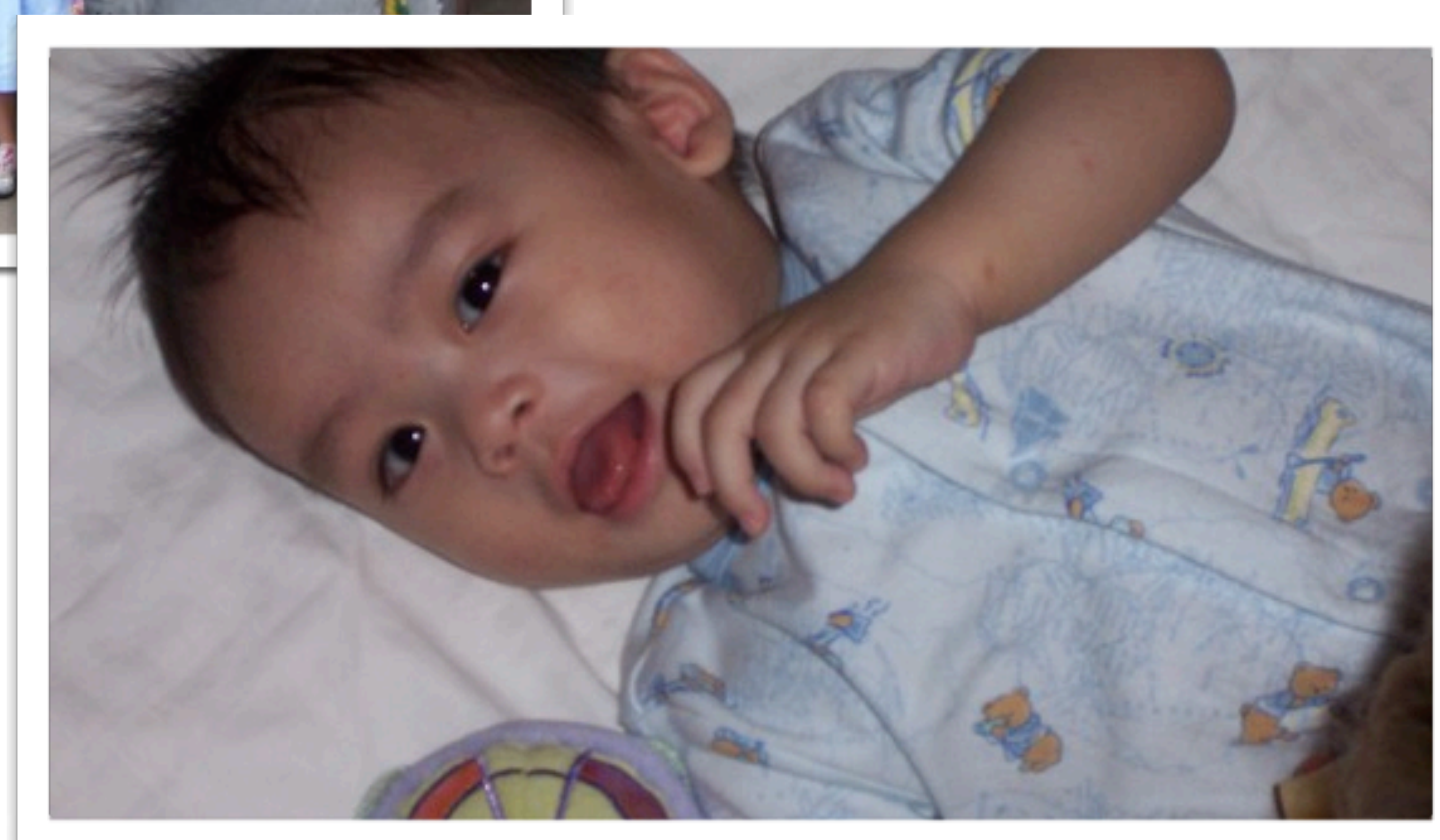
The Subtour LP for the Traveling Salesman Problem

David P. Williamson
Cornell University
March 6, 2012

Joint work with Frans Schalekamp and Anke van Zuylen

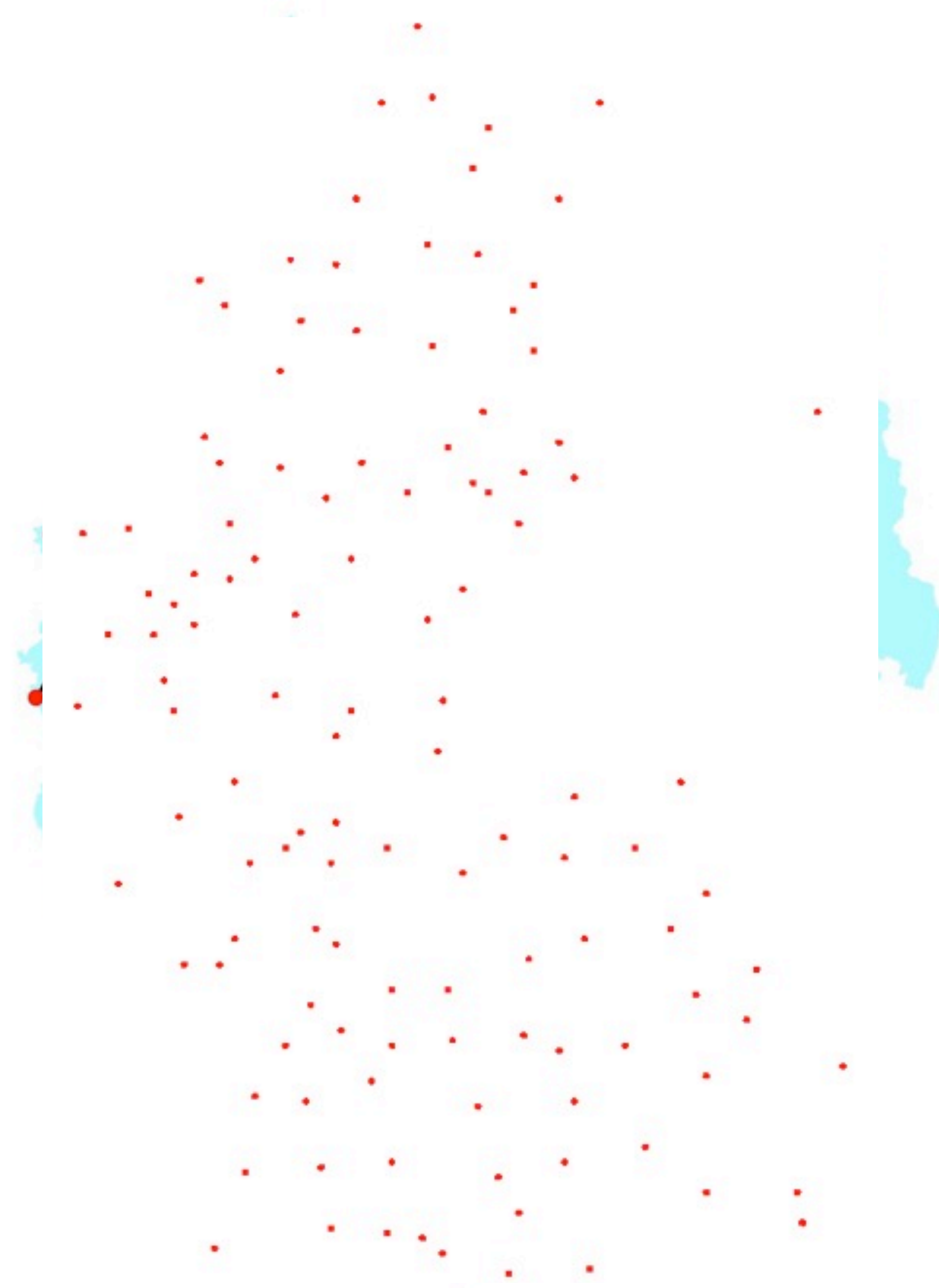

當明月坐湖心
洲有路通



The Traveling Salesman Problem

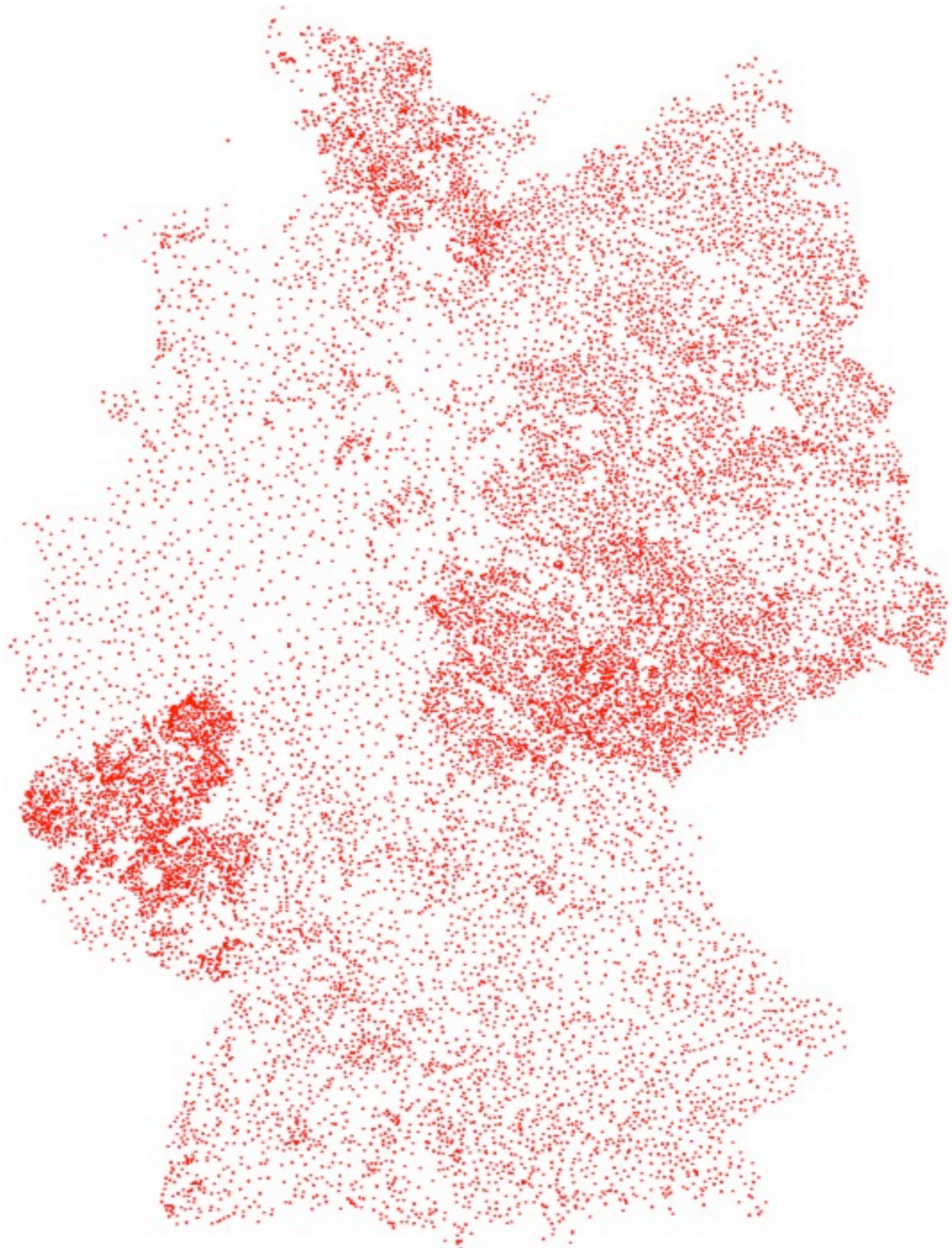
The most famous problem in discrete optimization: Given *n* cities and the cost *c(i,j)* of traveling from city *i* to city *j*, find a minimum-cost tour that visits each city exactly once.

We assume costs are symmetric (*c(i,j)*=*c(j,i)* for all i,j) and obey the triangle inequality (*c(i,j)* ≤ *c(i,k)* + *c(k,j)* for all i,j,k).

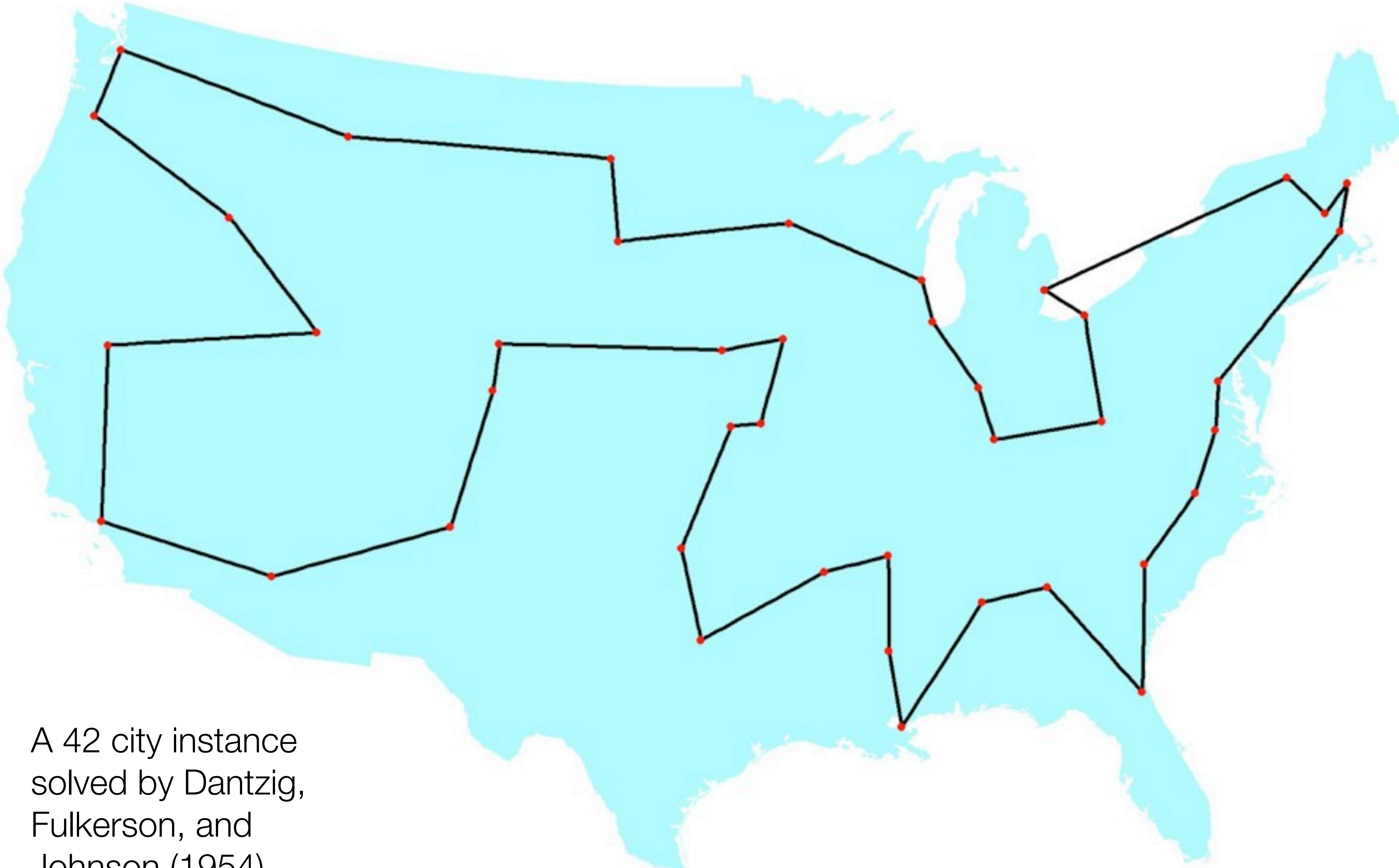


120 city tour of West Germany due to M. Grötschel (1977)

A 15112 city instance solved by Applegate, Bixby, Chvátal, and Cook (2001)



A 24978 city instance
from Sweden solved
by Applegate, Bixby,
Chvátal, Cook, and
Helsgaun (2004)



A 42 city instance solved by Dantzig, Fulkerson, and Johnson (1954)

The Dantzig-Fulkerson-Johnson Method

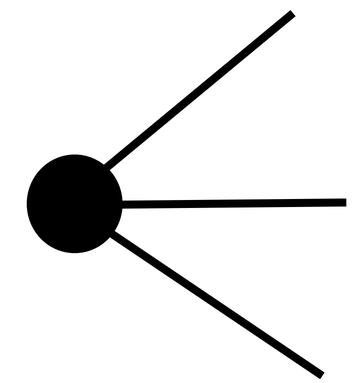
- G=(V,E) is a complete graph on *n* vertices
- c(e)=c(i,j) is the cost of traveling on edge e=(i,j)
- Solve linear programming (LP) relaxaion of the problem; if not integral, add additional constraint (cutting plane)
- x(e) is a decision variable indicating if edge e is used in the tour, $0 \leq x(e) \leq 1$

Subtour LP

$$\text{Minimize } \sum_{e \in E} c(e)x(e)$$

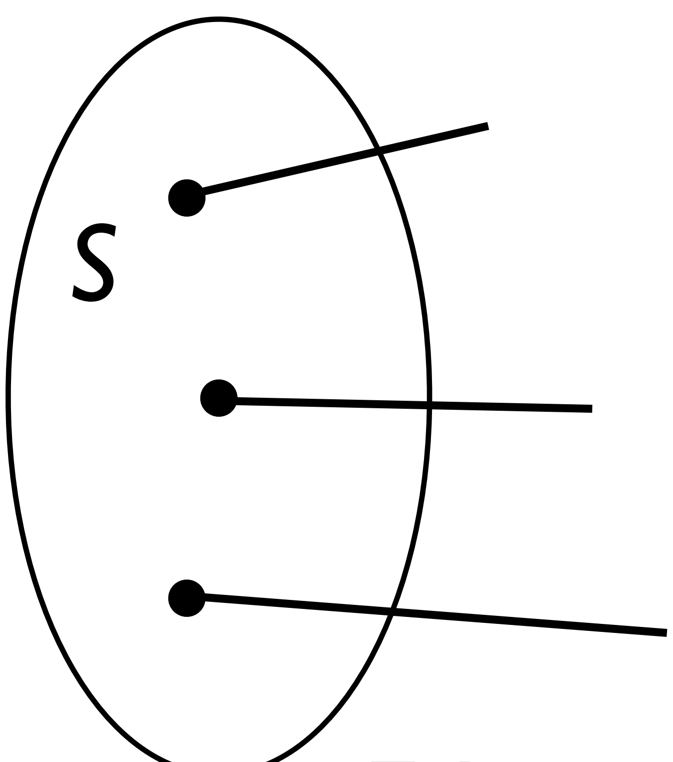
subject to

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$



$$\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subseteq V, |S| \geq 2$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$



Edges in the *cut* for *S*

How strong is the Subtour LP bound?

Johnson, McGeoch, and Rothberg (1996) and Johnson and McGeoch (2002) report experimentally that the Subtour LP is very close to the optimal.

Random Uniform Euclidean				TSPLIB			
Name	%Gap	Opttime	HKtime	Name	%Gap	Opttime	HKtime
E1k.0	0.77	1406	2.13	dsj1000	0.61	410	3.68
E1k.1	0.64	3855	2.15	pr1002	0.89	34	2.40
E1k.2	0.72	1211	2.02	si1032	0.08	25	11.32
E1k.3	0.62	956	1.92	u1060	0.65	571	3.62
E1k.4	0.69	330	1.69	vm1084	1.33	605	2.40
E1k.5	0.59	233	2.42	pcb1173	0.96	468	1.70
E1k.6	0.79	2940	1.67	d1291	1.18	27394	4.54
E1k.7	0.94	8003	1.95	rl1304	1.55	189	4.08
E1k.8	1.01	4347	1.65	rl1323	1.65	3742	4.49
E1k.9	0.61	189	2.14	nrw1379	0.43	578	2.40
E3k.0	0.71	533368	9.57	fl1400	1.74	1549	9.83
E3k.1	0.67	425631	10.54	u1432	0.29	224	2.42
E3k.2	0.74	342370	9.41	fl1577	1.66	6705	38.19
E3k.3	0.67	147135	10.30	d1655	0.94	263	6.51
E3k.4	0.73		8.07	vm1748	1.35	2224	4.43
Random Clustered Euclidean				u1817	0.90	449231	5.01
C1k.0	0.54	337	9.83	rl1889	1.55	10023	11.45
C1k.1	0.41	534	10.84	d2103	1.44	–	8.19
C1k.2	0.42	320	8.79	u2152	0.62	45205	8.10
C1k.3	0.53	214	7.63	u2319	0.02	7068	3.16
C1k.4	0.58	768	9.36	pr2392	1.22	117	5.75
C1k.5	0.58	139	9.29	pcb3038	0.81	80829	7.26
C1k.6	0.73	1247	7.07	fl3795	1.04	69886	123.66
C1k.7	0.58	449	13.24	fnl4461	0.55	–	12.47
C1k.8	0.34	140	10.40	rl5915	1.56	–	42.00
C1k.9	0.66	703	9.61	rl5934	1.38		56.15
C3k.0	0.62	16009	53.03	pla7397	0.58	–	55.42
C3k.1	0.61	17754	126.49	rl11849	1.02		102.41
C3k.2	0.70	18237	80.39	usa13509	0.66	–	120.20
C3k.3	0.57	6349	71.57	d15112	0.52		90.13
C3k.4	0.57	4845	44.02				
Random Matrices							
M1k.0	0.01	60	5.47	M3k.0	0.00	612	40.35
M1k.1	0.03	137	5.51	M3k.1	0.01	546	39.52
M1k.2	0.01	151	5.63	M10k.0	0.00	1377	367.84
M1k.3	0.01	169	5.26				

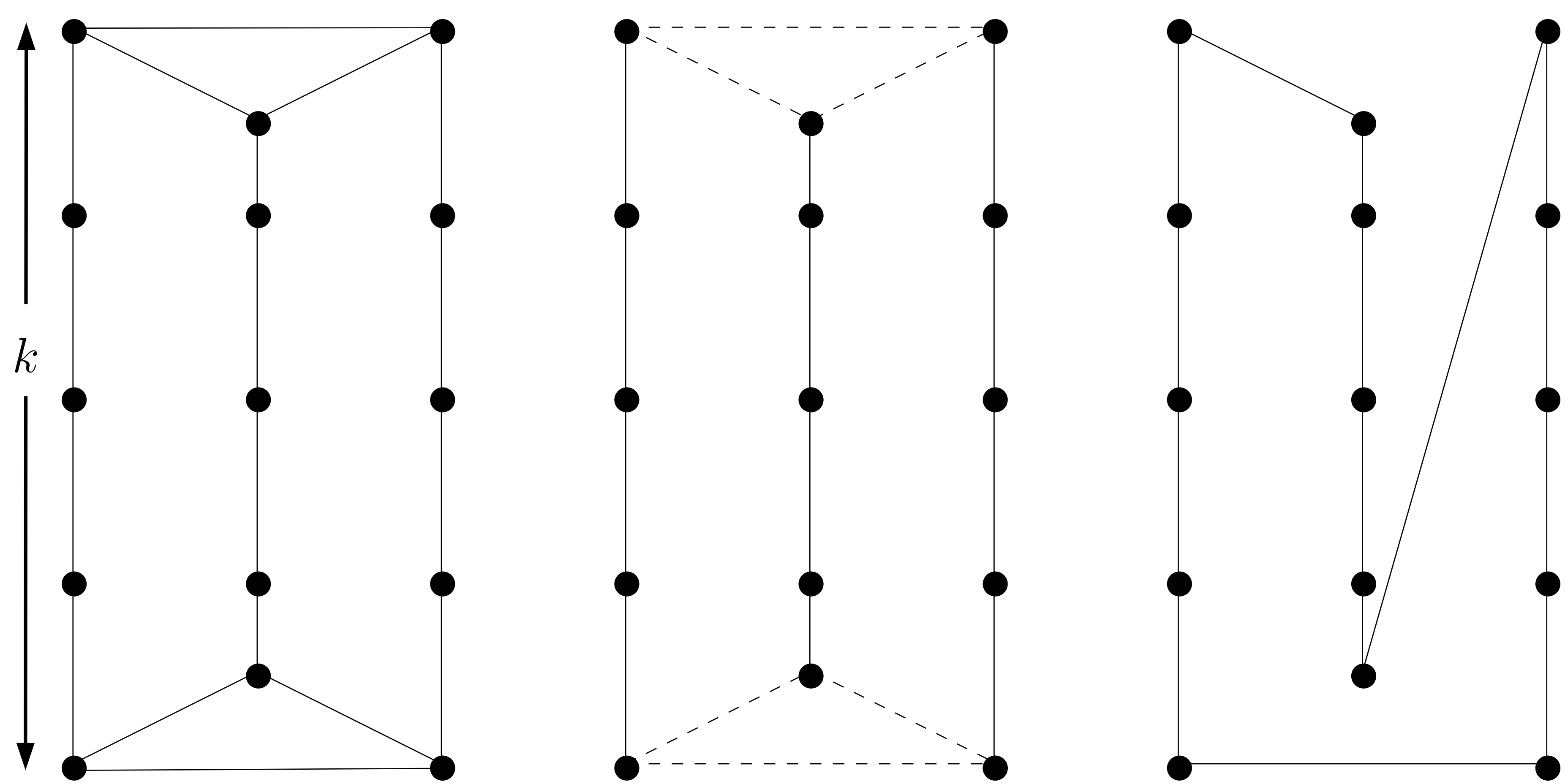
How strong is the Subtour LP bound?

- What about in theory?
- Define
  - SUBT(c) as the optimal value of the Subtour LP for costs c
  - OPT(c) as the length of the optimal tour for costs c
  - C_n is the set of all symmetric cost functions on *n* vertices that obey triangle inequality.
- Then the *integrality gap* of the Subtour LP is

$$\gamma \equiv \sup_n \gamma(n) \text{ where } \gamma(n) \equiv \sup_{c \in \mathcal{C}_n} \frac{OPT(c)}{SUBT(c)}$$

A lower bound

It's known that γ ≥ 4/3, where *c(i,j)* comes from the shortest *i-j* path distance in a graph G (*graphic TSP*).

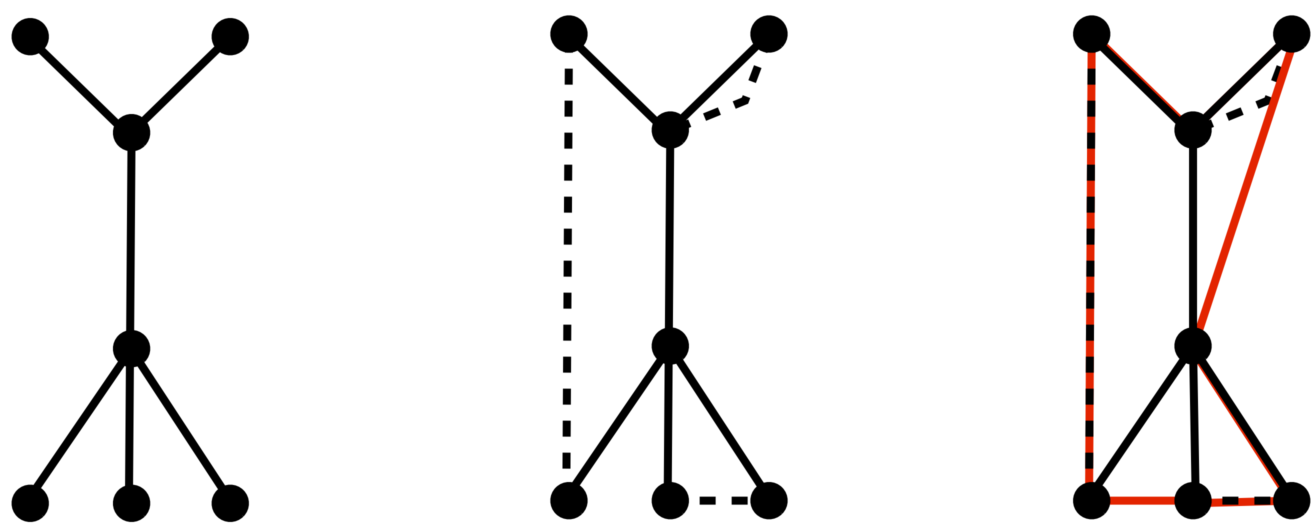


Graph G   LP soln   Opt tour

Christofides' Algorithm

Christofides (1976) shows how to compute a tour in polynomial time of cost 3/2 optimal:

- Compute a min-cost spanning tree
- Find a matching of odd-degree vertices
- Shortcut Eulerian traversal to tour



$\leq$ OPT(c)	+ $\leq$ 1/2 OPT(c)	$\leq$ 3/2 OPT(c)
$\leq$ SUBT(c)	+ $\leq$ 1/2 SUBT(c)	$\leq$ 3/2 SUBT(c)

Wolsey (1980),
Shmoys, W (1990)

An upper bound

- Therefore,

$$OPT(c) \leq \frac{3}{2} SUBT(c) \quad \Rightarrow \quad \gamma \leq \frac{OPT(c)}{SUBT(c)} \leq \frac{3}{2}$$

Recent results

- Some recent progress on graphic TSP (costs *c(i,j)* are the shortest *i*-*j* path distances in unweighted graph):
  - Boyd, Sitters, van der Ster, Stougie (2010): Gap is at most 4/3 if graph is cubic.
  - Oveis Gharan, Saberi, Singh (2010): Gap is at most 3/2 - ε for a constant ε > 0.
  - Mömke, Svensson (2011): Gap is at most 1.461.
  - Mömke, Svensson (2011): Gap is 4/3 if graph is subcubic (degree at most 3).
  - Mucha (2011): Gap is at most 13/9 ≈ 1.44.
  - Sebö, Vygen (2012): Gap is at most 7/5 = 1.4.

Current state

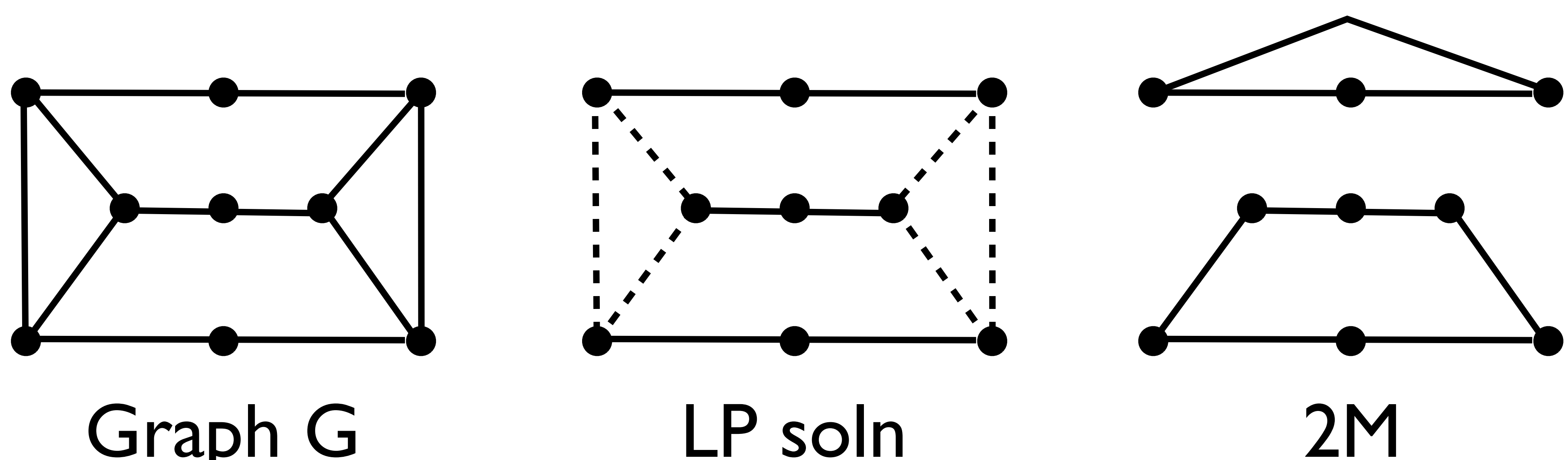
$$\frac{4}{3} \leq \gamma \leq \frac{3}{2}$$

- Conjecture (Goemans 1995, others): $\gamma = \frac{4}{3}$

More ignorance

We don’t even know the equivalent worst-case ratio between 2-matching costs 2M(c) and SUBT(c).

$$\mu \equiv \sup_n \mu(n) \text{ where } \mu(n) \equiv \sup_{c \in \mathcal{C}_n} \frac{2M(c)}{SUBT(c)}$$



More ignorance

We don't even know the equivalent worst-case ratio between 2-matching costs 2M(c) and SUBT(c).

$$\mu \equiv \sup_n \mu(n) \text{ where } \mu(n) \equiv \sup_{c \in \mathcal{C}_n} \frac{2M(c)}{SUBT(c)}$$

Then all we know is that

$$\frac{10}{9} \leq \mu \leq \frac{4}{3} \text{ (Boyd, Carr 1999)}$$

Conjecture (Boyd, Carr 2011): $\mu = \frac{10}{9}$

Our contributions

- We can prove the Boyd-Carr conjecture.

Outline

- Warm up: $\mu \leq 4/3$ under a certain condition.
- $\mu \leq 10/9$.
- Some conjectures.

~~Subtour LP~~

Fractional 2-matching LP

$$\text{Minimize } \sum_{e \in E} c(e)x(e)$$

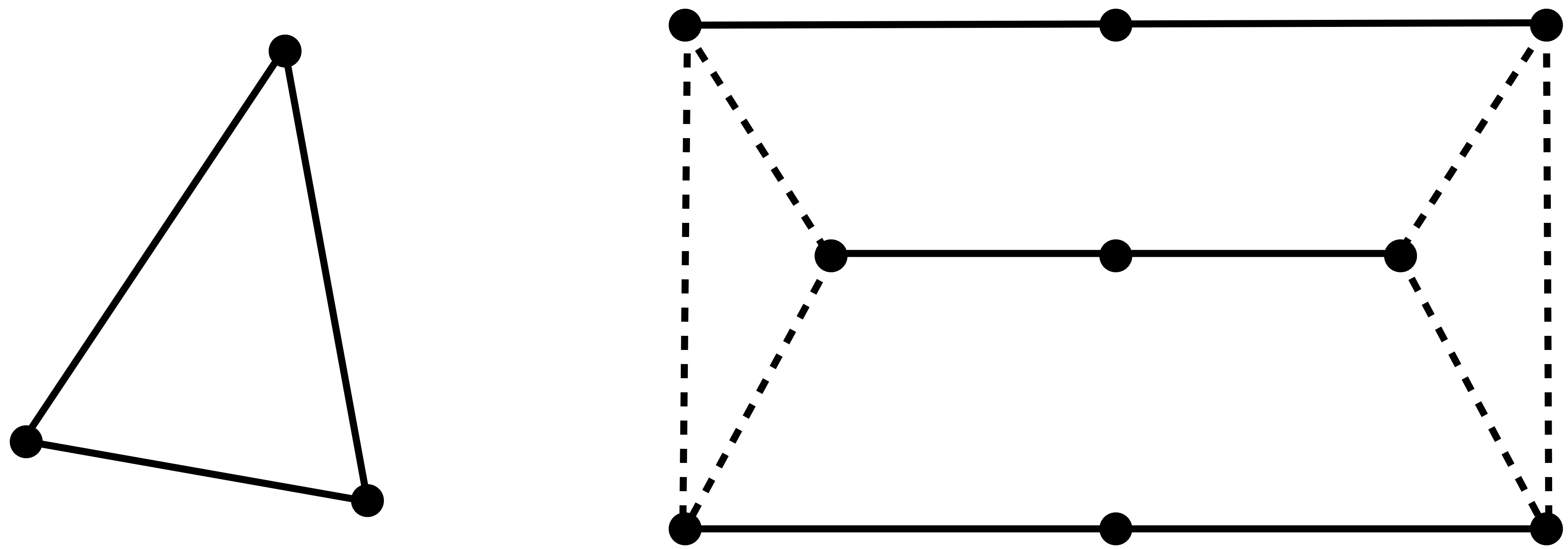
subject to

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$

$$\cancel{\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subseteq V, |S| \geq 2}$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$

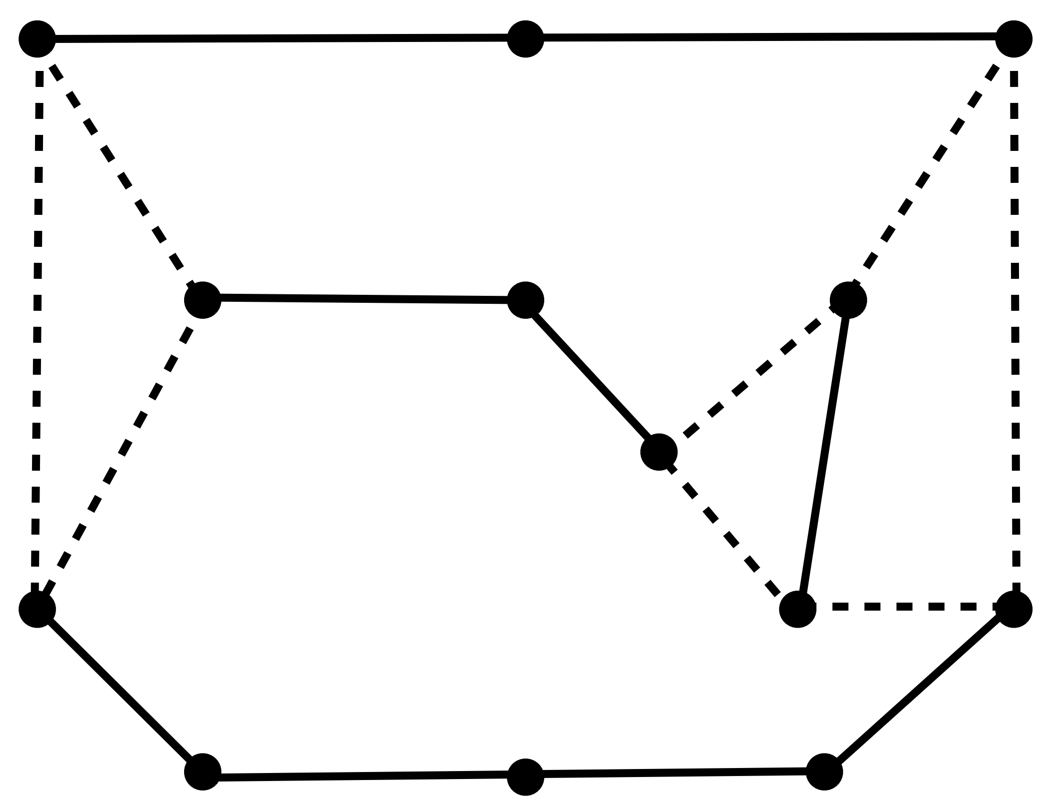
Fractional 2-matchings



Basic solutions to LP have components that are cycles of size at least 3 with x(e)=1 or odd cycles with x(e)=1/2 connected by paths with x(e)=1

Assumptions and terms

For now: assume optimal fractional 2-matching is feasible for the Subtour LP

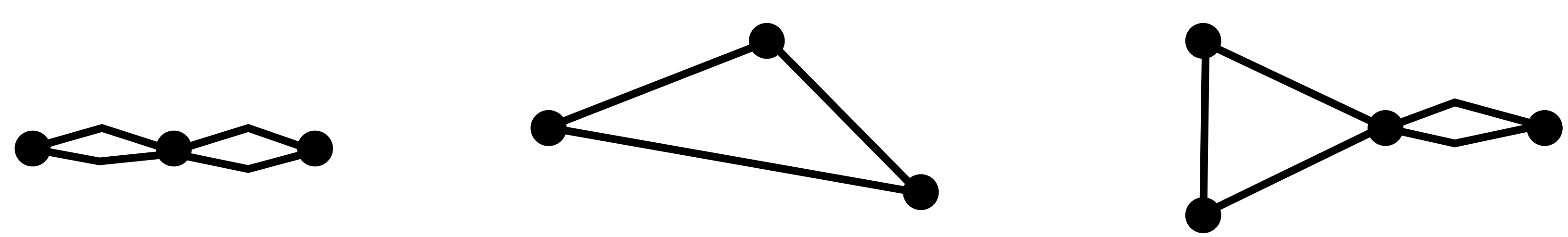


Path edge $x(e)=1$

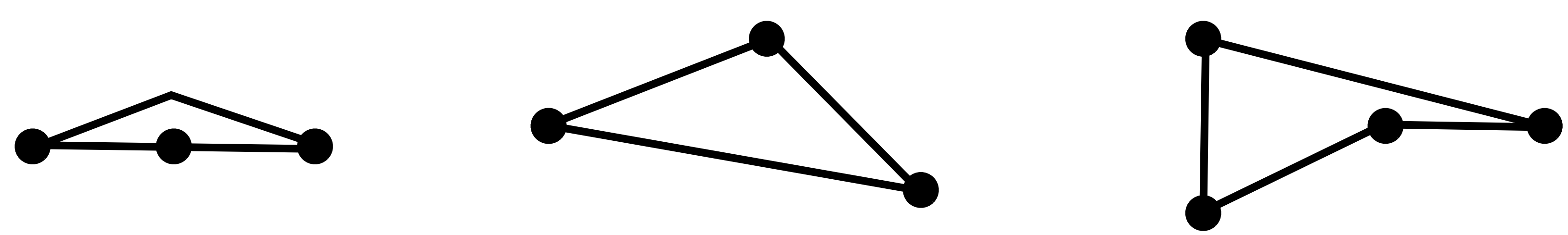
Cycle edge $x(e)=1/2$

The strategy

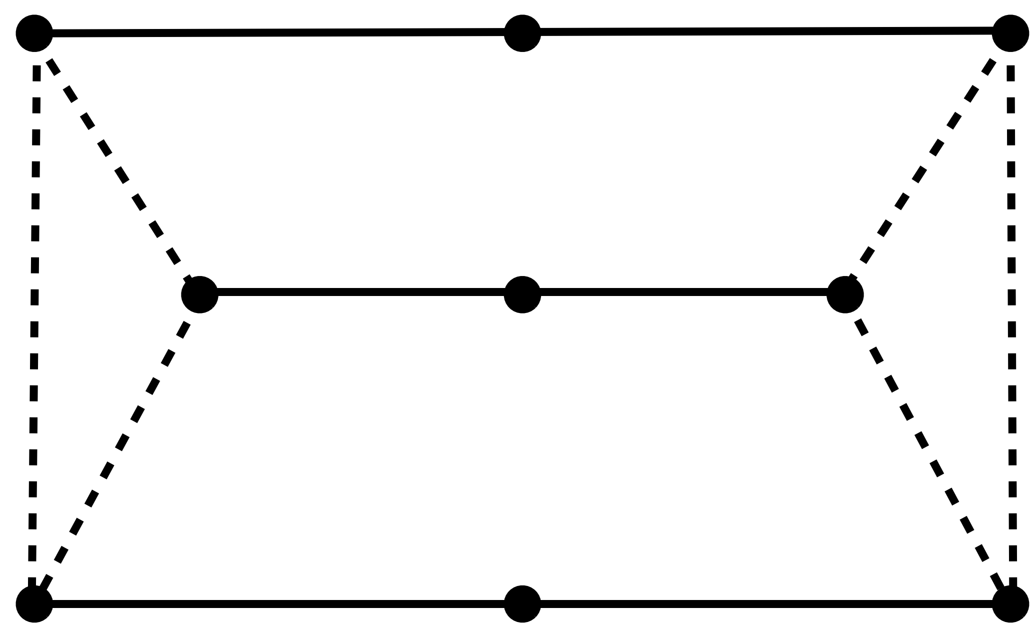
- Start with an optimal fractional 2-matching; this gives a lower bound on the Subtour LP.
- Add a low-cost set of edges to create a *graphical* 2-matching: each vertex has degree 2 or 4; each component has size at least 3; each edge has 0, 1, or 2 copies.



- "Shortcut" the graphical 2-matching to a 2-matching.

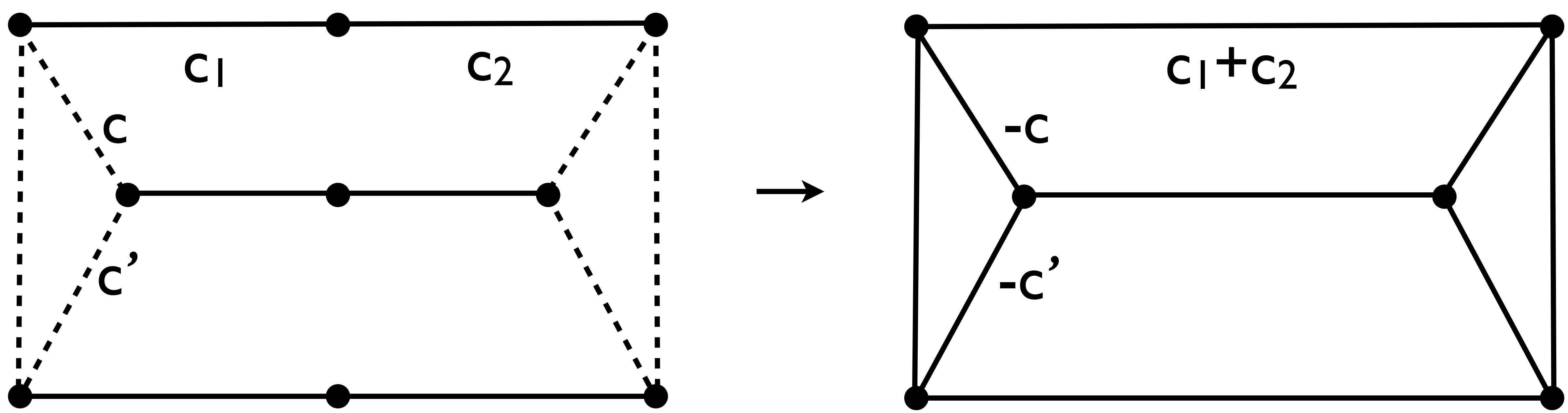


For fractional 2-matchings feasible for Subtour LP, we show that we can get a graphical 2-matching with a 4/3 increase in cost.



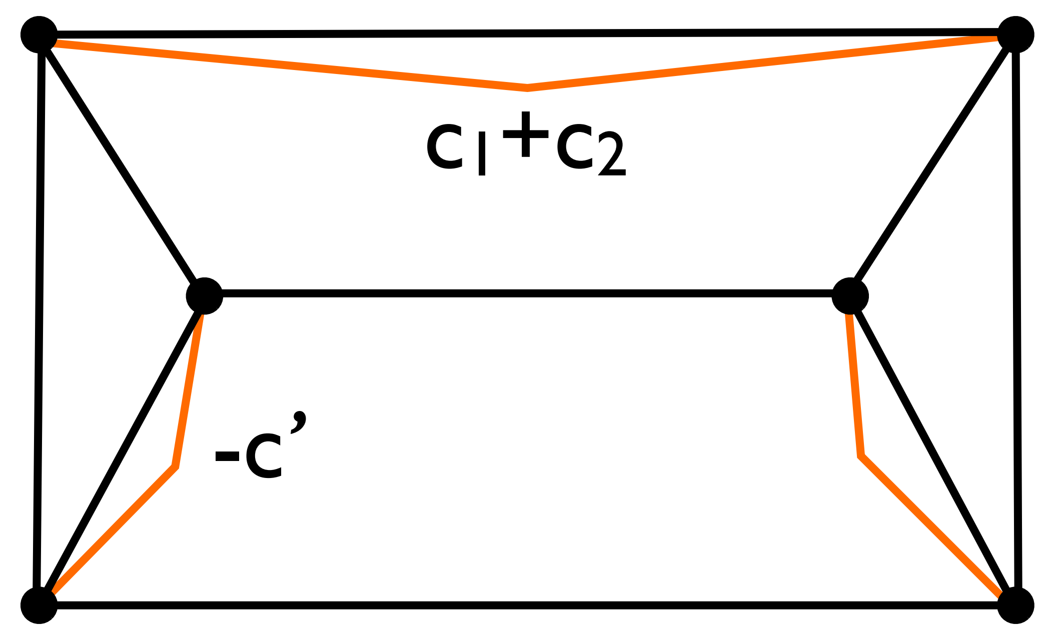
2M $\leq$ Graphical 2M $\leq$ 4/3 Fractional 2M $\leq$ 4/3 Subtour

Create new graph by replacing path edges with a single edge of cost equal to the path, cycle edges with negations of their cost.

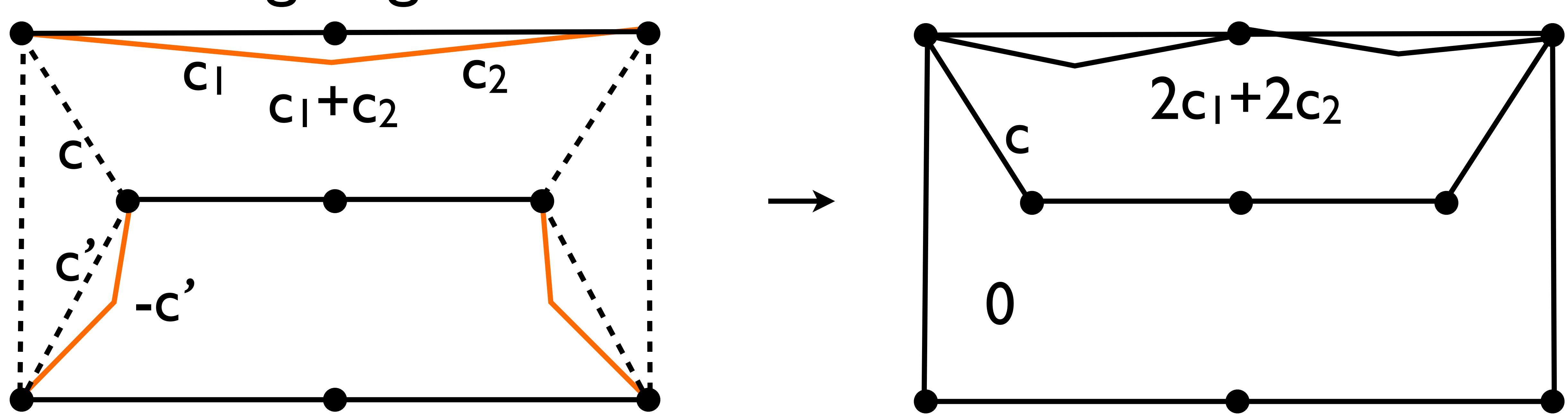


New graph is cubic and 2-edge connected.

Compute a min-cost perfect matching in new graph.

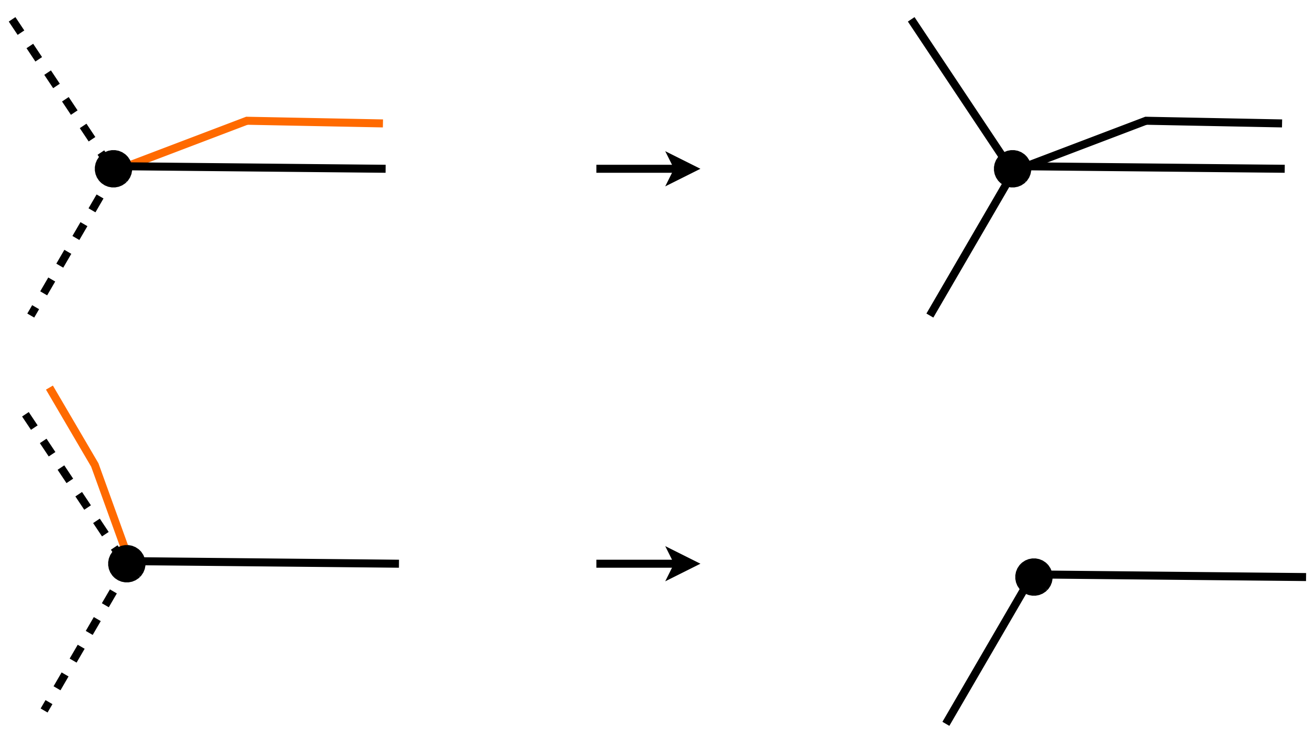


In the fractional 2-matching, double any path edge in matching, remove any cycle edge. Cost is paths + cycles + matching edges.



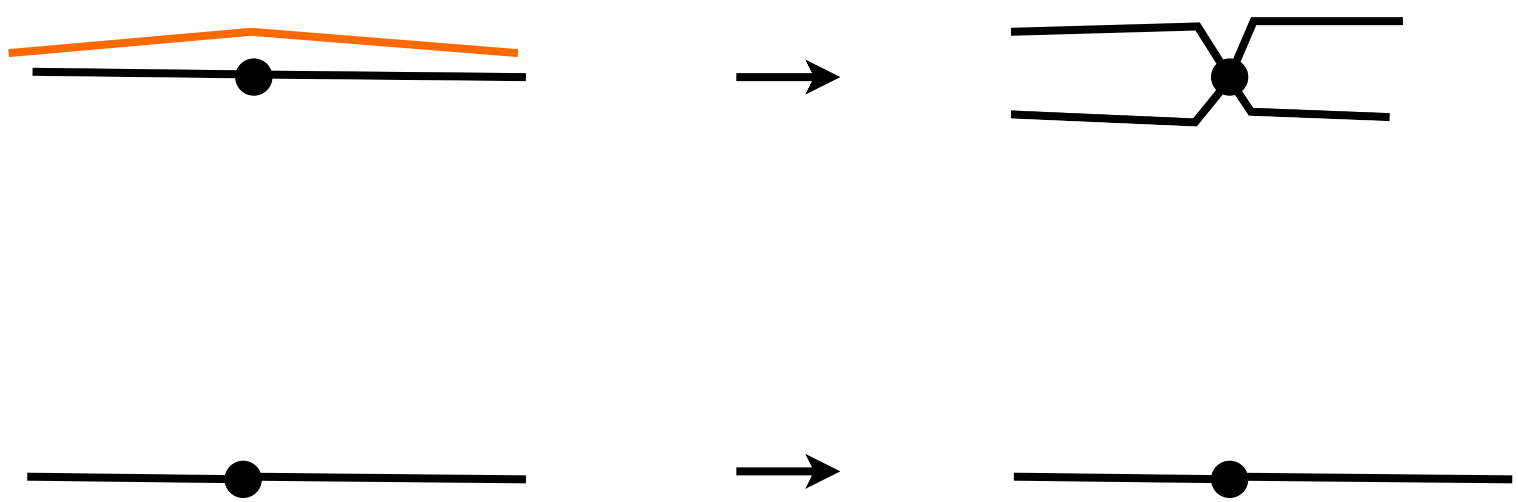
Why this works

For any given node on the cycle, either its associated path edge is in the matching or one of the two cycle edges.



Why this works

For any given node on the path, either its associated path edge is in the matching or not.



Bounding the cost

- P = total cost of all path edges
- C = total cost all cycle edges
- So fractional 2-matching costs P + C/2
- Claim: Perfect matching in the new graph costs at most 1/3 the cost of all its edges, so at most 1/3(P - C)

Bounding the cost

- Since the graphical 2-matching costs at most P + C + matching, it costs at most

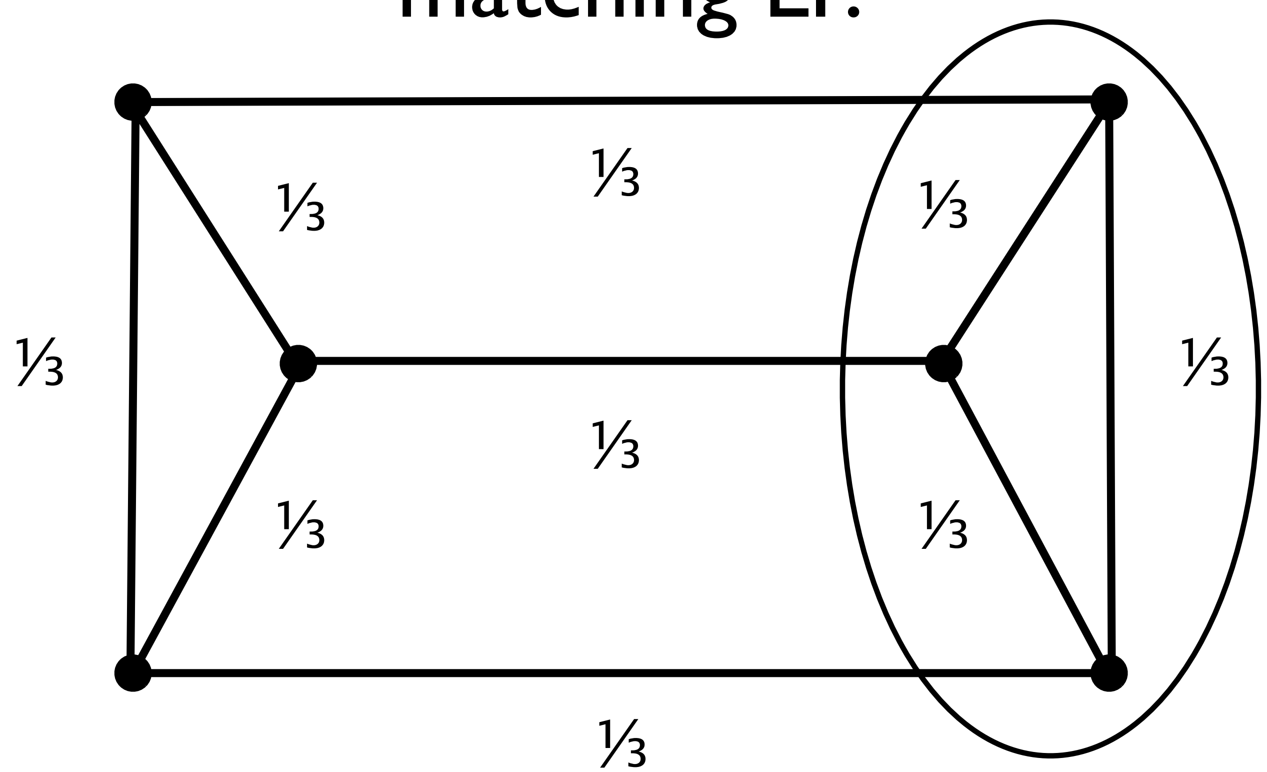
$$P + C + \frac{1}{3}(P - C) = \frac{4}{3}P + \frac{2}{3}C = \frac{4}{3}\left(P + \frac{1}{2}C\right)$$

2M ≤ Graphical 2M ≤ 4/3 Fractional 2M

≤ 4/3 Subtour

Matching cost

- Naddef and Pulleyblank (1981): Any cubic, 2-edge-connected, weighted graph has a perfect matching of cost at most a third of the sum of the edge weights.
- Proof: Set z(e)=1/3 for all e∈E, then feasible for matching LP.

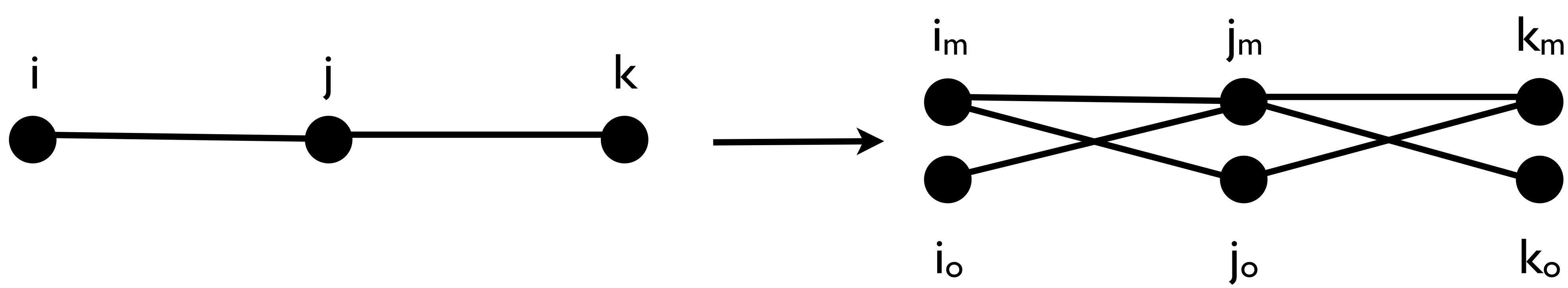


$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} c(e) z(e) \\
\text{subject to} \quad & \sum_{e \in \delta(v)} z(e) = 1 \qquad \forall v \in V \\
& \sum_{e \in \delta(S)} z(e) \geq 1 \qquad \forall S \subset V, |S| \text{ odd}
\end{aligned}
$$

By parity argument any odd-sized set S must have odd |δ(S)|.

Another route

- With more work, can prove $\mu \leq 10/9$ under same condition.
- To drop the assumption that fractional 2-matching is feasible for Subtour LP, we give a polyhedral formulation for graphical 2-matchings.
- Reduce to "2 matching with optional nodes":
  - *Mandatory* copy of node i_m must have degree 2 in solution
  - *Optional* copy of node i_o can have degree 2 or 0
  - No edges between optional copies implies each component has size at least three.



The formulation

$$\sum_{e \in \delta(i_m)} y(e) = 2 \qquad \forall i_m$$

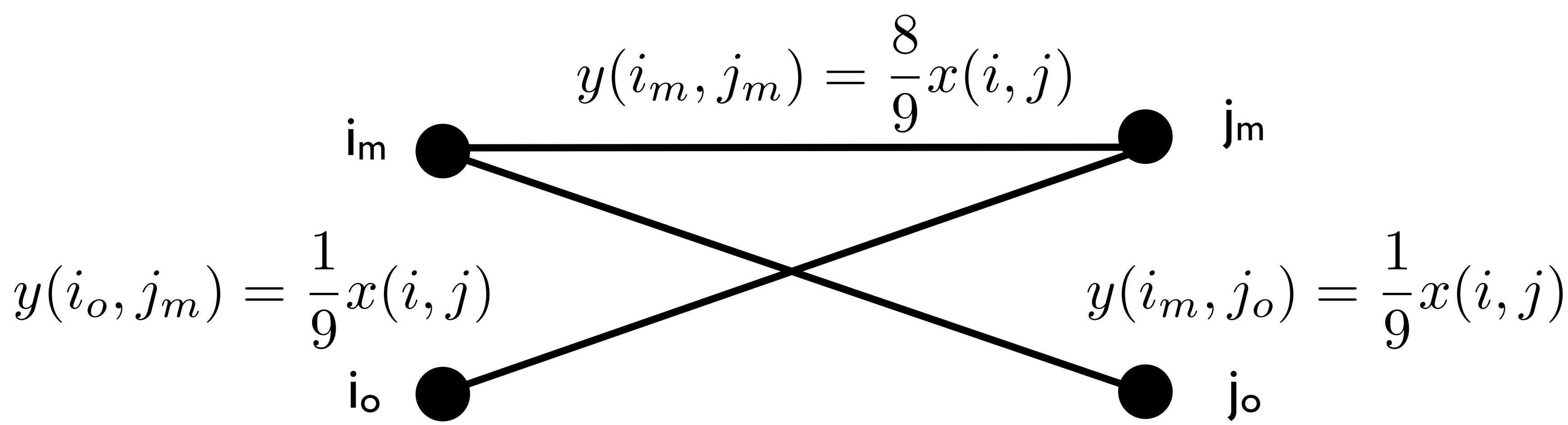
$$\sum_{e \in \delta(i_o)} y(e) \leq 2 \qquad \forall i_o$$

$$\sum_{e \in \delta(S)-F} y(e) + |F| - \sum_{e \in F} y(e) \geq 1 \qquad \forall S \subseteq V, F \subseteq \delta(S), F \text{ matching}, |F| \text{ odd}$$

$$0 \leq y(e) \leq 1 \qquad \forall e \in E$$

Showing that μ ≤10/9

Given Subtour LP soln x, set



$$\sum_{e \in \delta(i_m)} y(e) = 2 \qquad \forall i_m$$

$$\sum_{e \in \delta(i_o)} y(e) \leq 2 \qquad \forall i_o$$

$$\sum_{e \in \delta(S)-F} y(e) + |F| - \sum_{e \in F} y(e) \geq 1$$

$$\forall S \subseteq V, F \subseteq \delta(S), F \text{ matching}, |F| \text{ odd}$$

$$0 \leq y(e) \leq 1 \qquad \forall e \in E$$

$$\text{Minimize } \sum_{e \in E} c(e)x(e)$$

subject to

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$

$$\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subseteq V, |S| \geq 2$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$

Edmonds (1967)

traveling saleman problem [cf. 4]. I conjecture that there is no good algorithm for the traveling saleman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know.

A good algorithm is known for finding, in any graph

Conclusion

- Observation: The worst-case ratio for 2-matchings to the Subtour LP occurs when the optimal subtour solution is a fractional 2-matching.
- Conjecture: The worst-case ratio for the Subtour LP integrality gap occurs when the optimal subtour solution is a fractional 2-matching.
- Not sure what we can prove if the conjecture is true!

"Theory is when we understand everything, but nothing works.

Practice is when everything works, but we don't understand why.

At this station, theory and practice are united, so that nothing works and no one understands why."

Thanks for
your attention.