The Subtour LP for the Traveling Salesman Problem

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April 8, 2015

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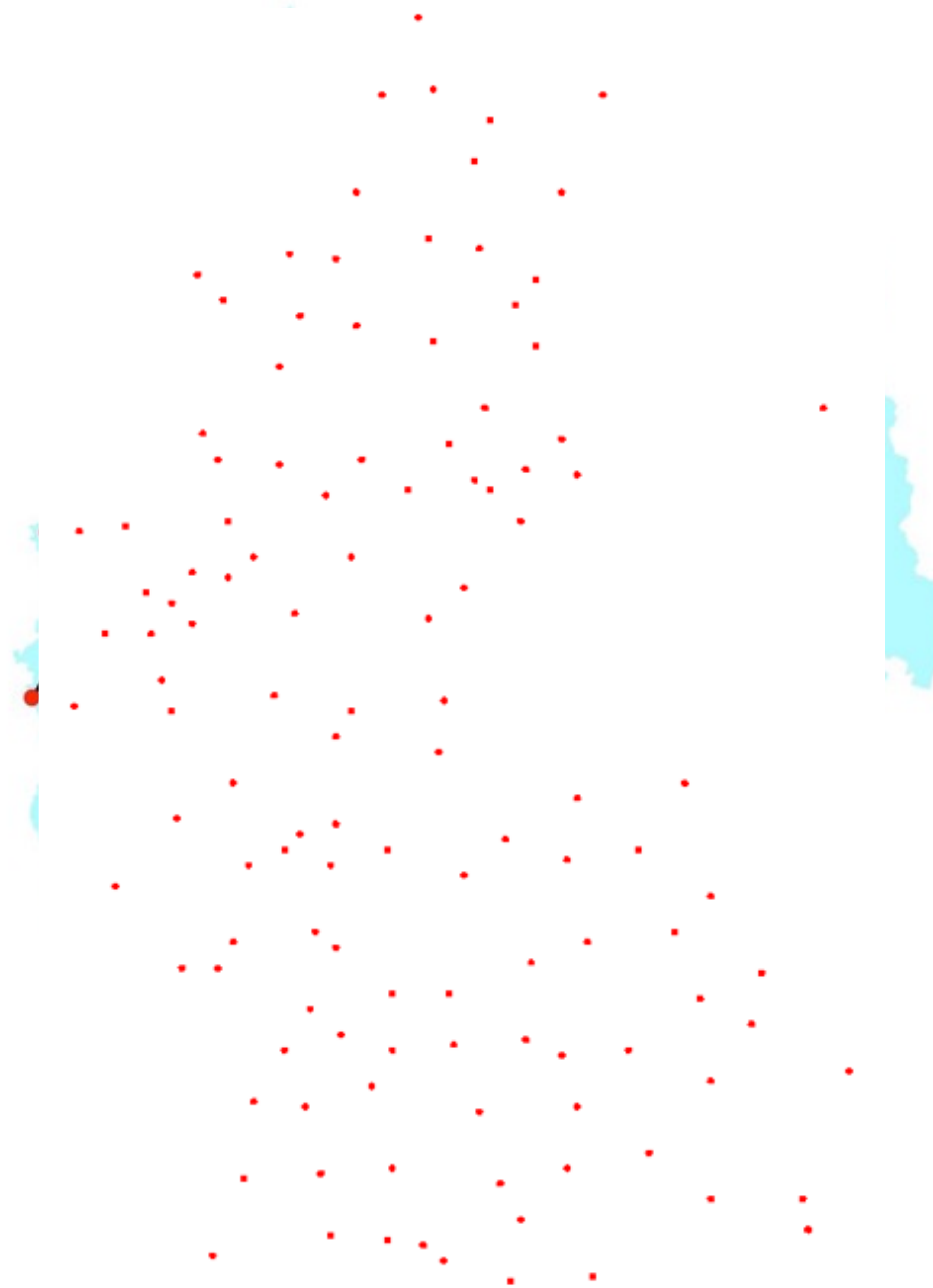
Outline

- A brief intro to the TSP
- A standard TSP linear program: the Subtour LP
    - Experimental analysis
    - Theoretical analysis: an outstanding open question
- A related question: the Boyd-Carr conjecture and its proof
- Some conjectures and more experiments

The Traveling Salesman Problem

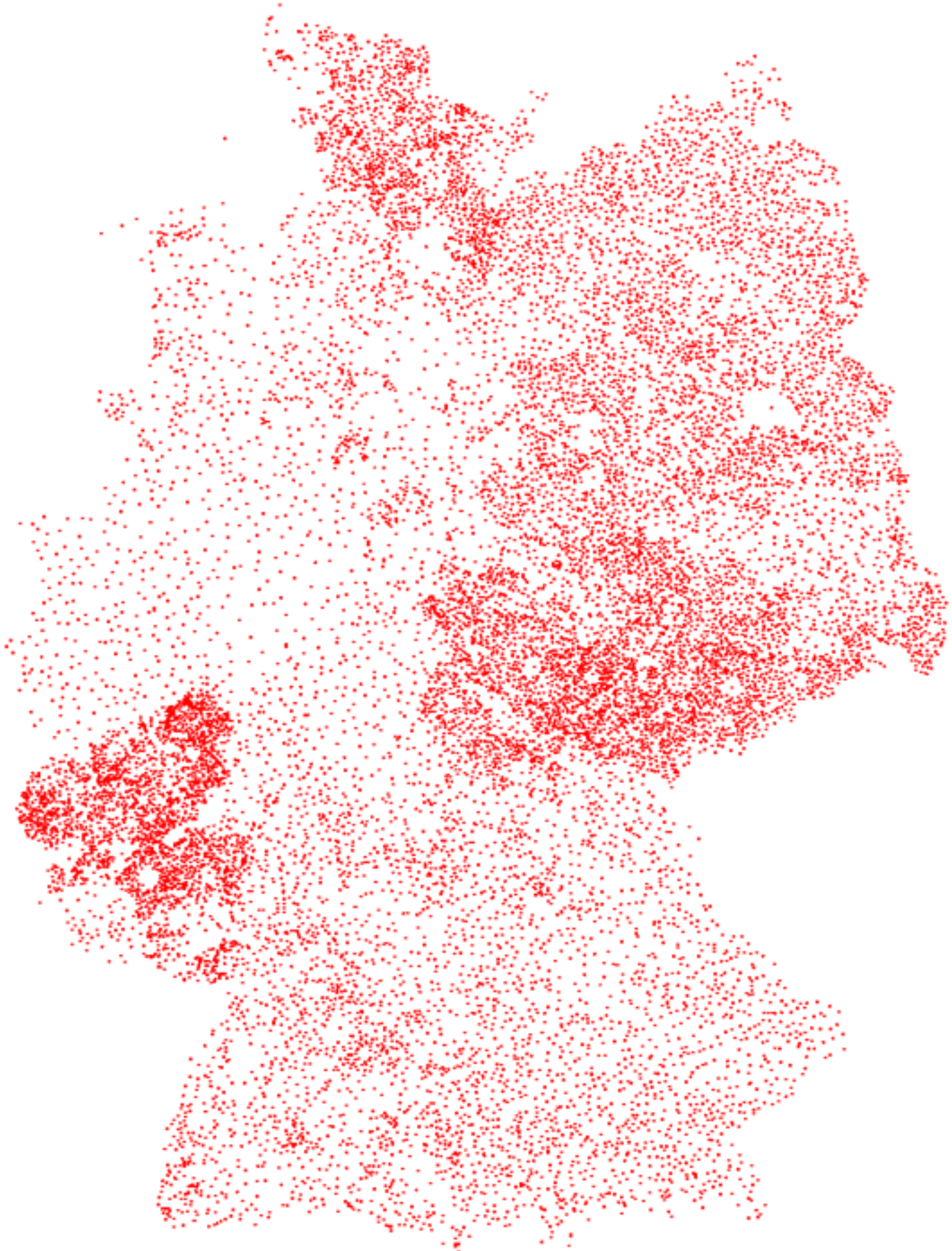
The most famous problem in discrete optimization: Given *n* cities and the cost *c(i,j)* of traveling from city *i* to city *j*, find a minimum-cost tour that visits each city exactly once.

We assume costs are symmetric (*c(i,j)*=*c(j,i)* for all i,j) and obey the triangle inequality (*c(i,j)* ≤ *c(i,k)* + *c(k,j)* for all i,j,k).

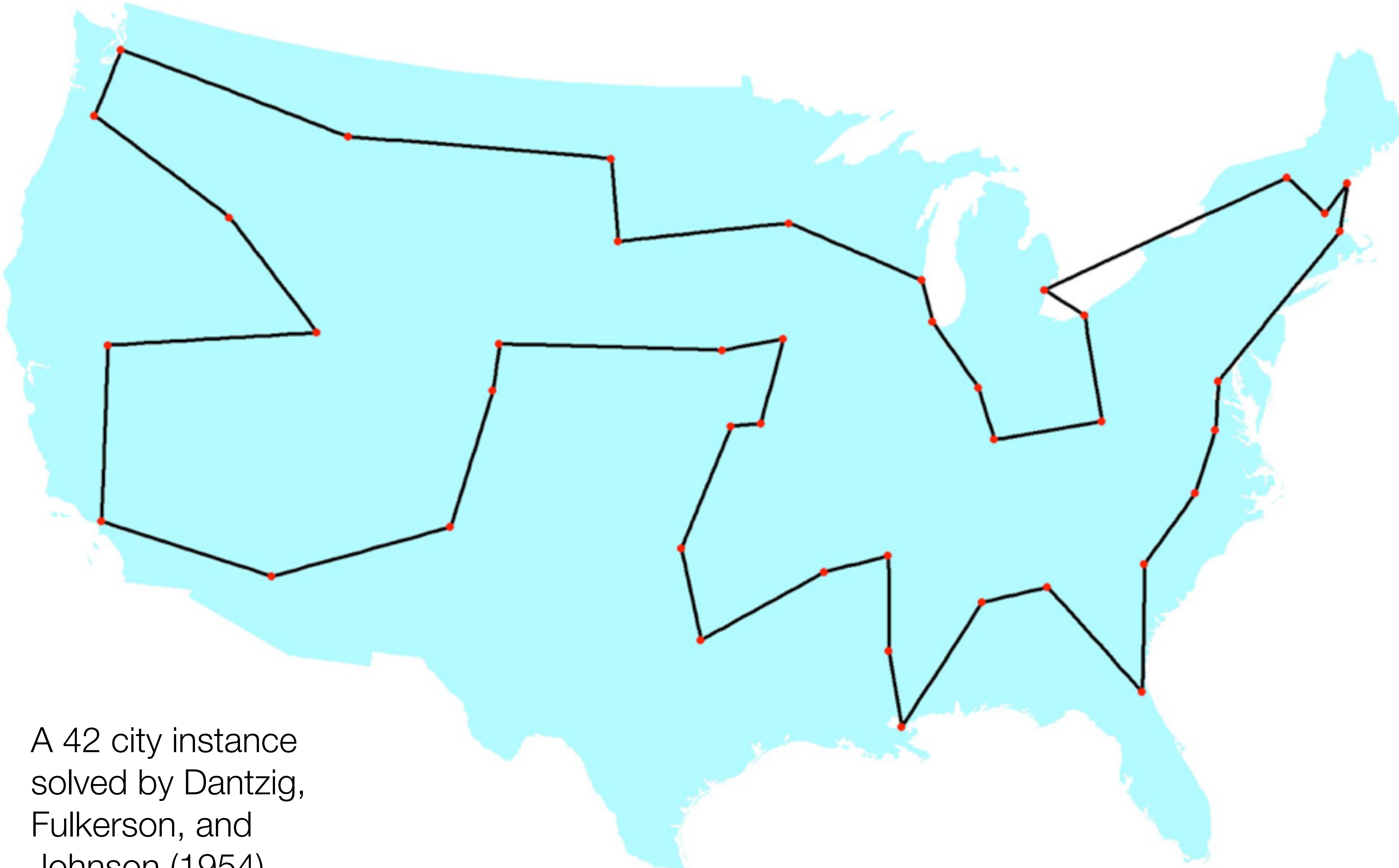


120 city tour of West Germany due to M. Grötschel (1977)

A 15112 city instance solved by Applegate, Bixby, Chvátal, and Cook (2001)



A 24978 city instance from Sweden solved by Applegate, Bixby, Chvátal, Cook, and Helsgaun (2004)



A 42 city instance solved by Dantzig, Fulkerson, and Johnson (1954)

The Dantzig-Fulkerson-Johnson Method

- $G=(V,E)$ is a complete graph on $|V| = n$ vertices
- $c(e)=c(i,j)$ is the cost of traveling on edge $e=(i,j)$
- $x(e)$ is a decision variable indicating if edge e is used in the tour, $0 \leq x(e) \leq 1$
- Solve linear program; if $x(e)$ forms integer tour, stop, else find a *cutting plane*

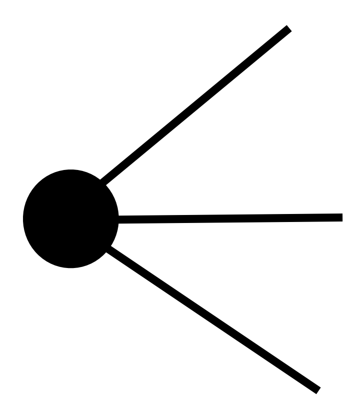
The linear program

$$\text{Minimize } \sum_{e \in E} c(e)x(e)$$

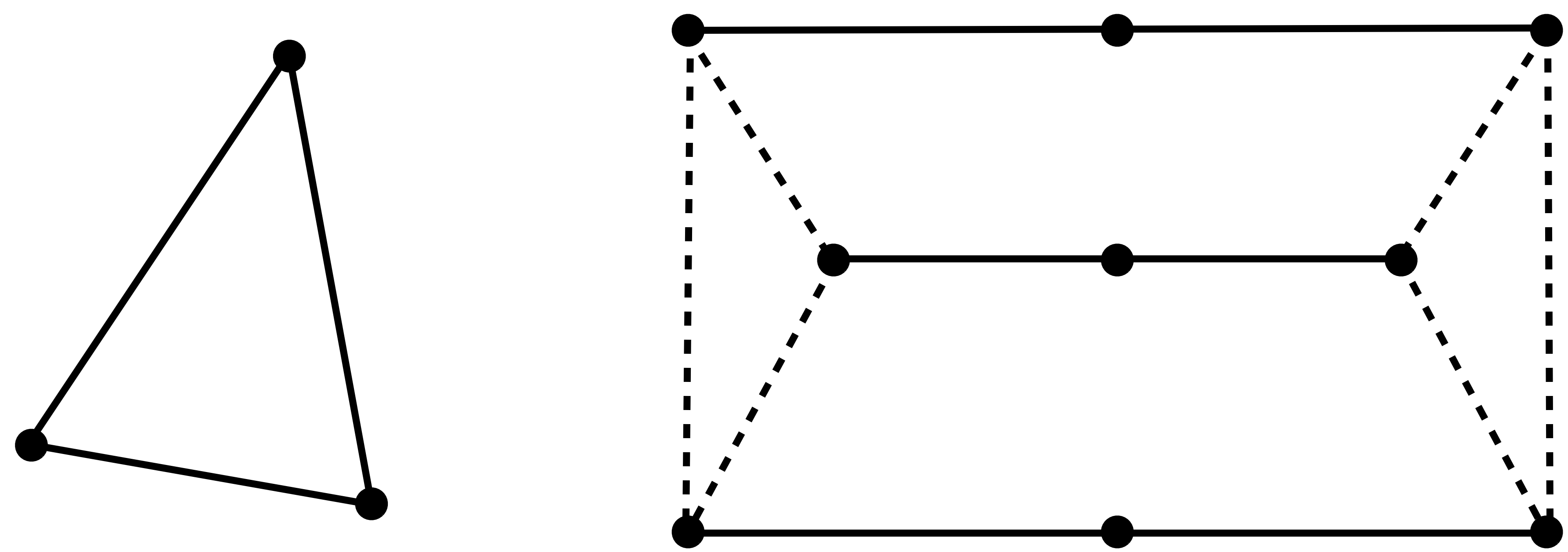
subject to

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$

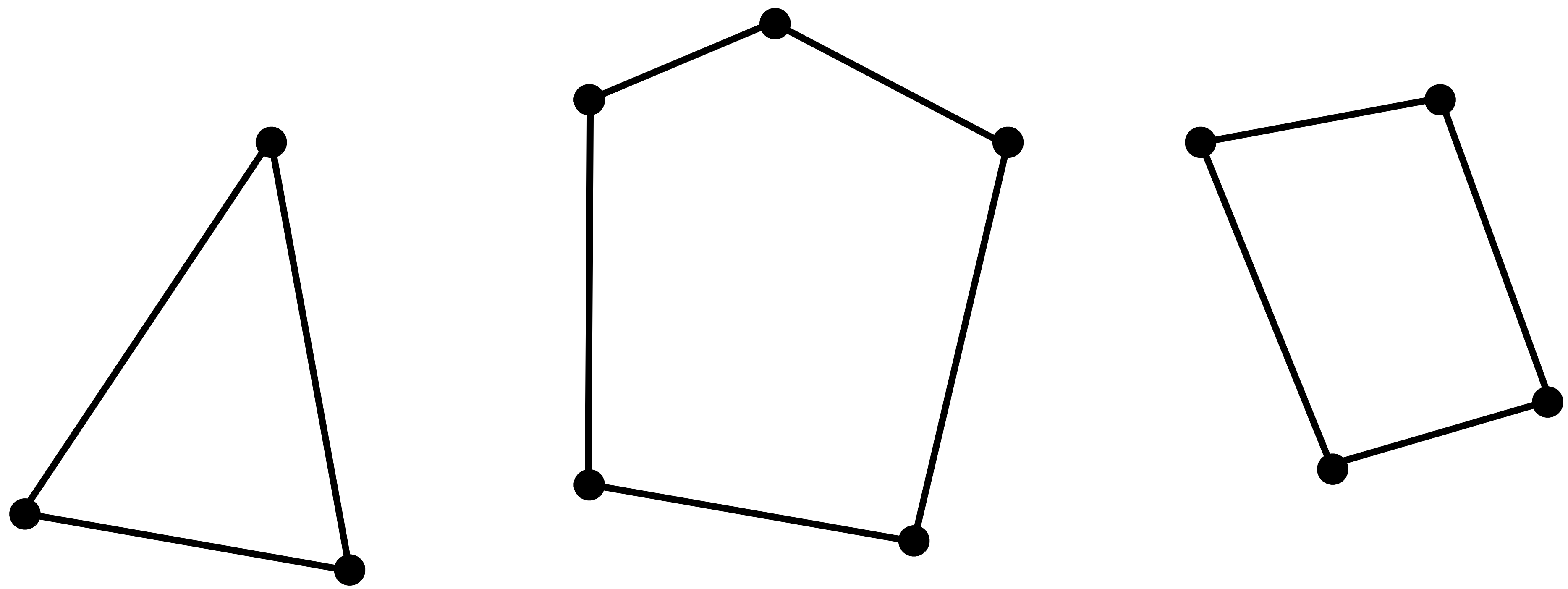


Fractional 2-matchings



Fractional (basic) solutions have components that are cycles of size at least 3 with x(e)=1 or odd cycles with x(e)=1/2 connected by paths with x(e)=1

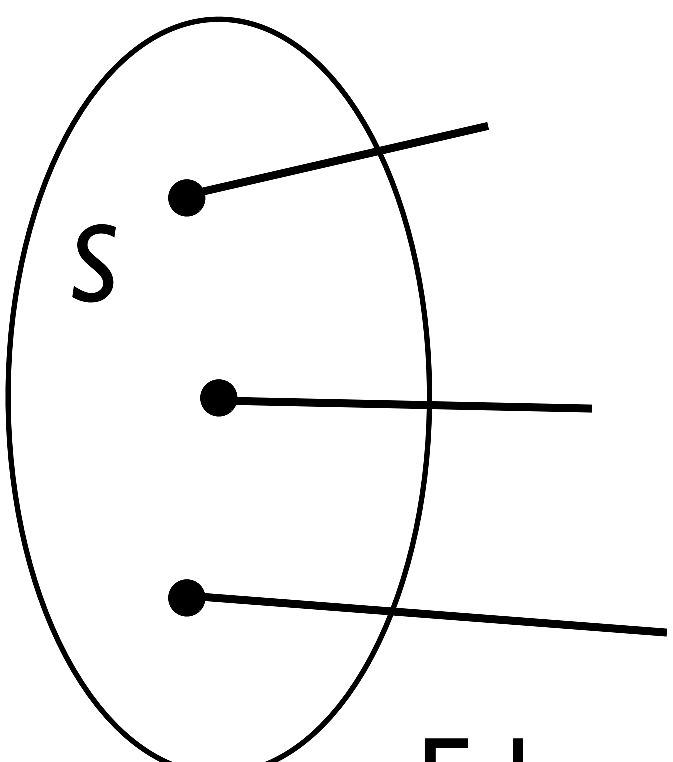
2-matchings



Integer solutions have components with cycles of size at least 3; sometimes called *subtours*

“Loop conditions”

Dantzig, Fulkerson, and Johnson added constraints to eliminate subtours as they occurred; these now called “subtour elimination constraints”.



$$\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subseteq V, |S| \geq 2$$

Edges in the *cut* for *S*

Subtour LP

$$\text{Minimize } \sum_{e \in E} c(e)x(e)$$

subject to

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$

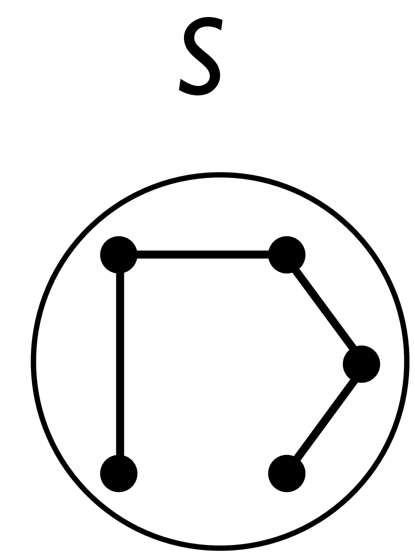
$$\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subseteq V, |S| \geq 2$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$

Equivalent constraints

Equivalently can write subtour elimination constraints to express no cycles in any strict subset:

$$\sum_{e \in E(S)} x(e) \leq |S| - 1 \qquad \forall S \subset V, |S| \geq 2$$



Subtour LP

$$\text{Minimize} \ \sum_{e \in E} c(e)x(e)$$

$$\text{subject to}$$

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$

$$\sum_{e \in E(S)} x(e) \leq |S| - 1 \qquad \forall S \subset V, |S| \geq 2$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$

How strong is the Subtour LP bound?

Johnson, McGeoch, and Rothberg (1996) and Johnson and McGeoch (2002) report experimentally that the Subtour LP is very close to the optimal.

Random Uniform Euclidean				TSPLIB			
Name	%Gap	Opttime	HKtime	Name	%Gap	Opttime	HKtime
E1k.0	0.77	1406	2.13	dsj1000	0.61	410	3.68
E1k.1	0.64	3855	2.15	pr1002	0.89	34	2.40
E1k.2	0.72	1211	2.02	si1032	0.08	25	11.32
E1k.3	0.62	956	1.92	u1060	0.65	571	3.62
E1k.4	0.69	330	1.69	vm1084	1.33	605	2.40
E1k.5	0.59	233	2.42	pcb1173	0.96	468	1.70
E1k.6	0.79	2940	1.67	d1291	1.18	27394	4.54
E1k.7	0.94	8003	1.95	rl1304	1.55	189	4.08
E1k.8	1.01	4347	1.65	rl1323	1.65	3742	4.49
E1k.9	0.61	189	2.14	nrw1379	0.43	578	2.40
E3k.0	0.71	533368	9.57	fl1400	1.74	1549	9.83
E3k.1	0.67	425631	10.54	u1432	0.29	224	2.42
E3k.2	0.74	342370	9.41	fl1577	1.66	6705	38.19
E3k.3	0.67	147135	10.30	d1655	0.94	263	6.51
E3k.4	0.73		8.07	vm1748	1.35	2224	4.43
Random Clustered Euclidean				u1817	0.90	449231	5.01
C1k.0	0.54	337	9.83	rl1889	1.55	10023	11.45
C1k.1	0.41	534	10.84	d2103	1.44	–	8.19
C1k.2	0.42	320	8.79	u2152	0.62	45205	8.10
C1k.3	0.53	214	7.63	u2319	0.02	7068	3.16
C1k.4	0.58	768	9.36	pr2392	1.22	117	5.75
C1k.5	0.58	139	9.29	pcb3038	0.81	80829	7.26
C1k.6	0.73	1247	7.07	fl3795	1.04	69886	123.66
C1k.7	0.58	449	13.24	fnl4461	0.55	–	12.47
C1k.8	0.34	140	10.40	rl5915	1.56	–	42.00
C1k.9	0.66	703	9.61	rl5934	1.38		56.15
C3k.0	0.62	16009	53.03	pla7397	0.58	–	55.42
C3k.1	0.61	17754	126.49	rl11849	1.02		102.41
C3k.2	0.70	18237	80.39	usa13509	0.66	–	120.20
C3k.3	0.57	6349	71.57	d15112	0.52		90.13
C3k.4	0.57	4845	44.02				
Random Matrices							
M1k.0	0.01	60	5.47	M3k.0	0.00	612	40.35
M1k.1	0.03	137	5.51	M3k.1	0.01	546	39.52
M1k.2	0.01	151	5.63	M10k.0	0.00	1377	367.84
M1k.3	0.01	169	5.26				

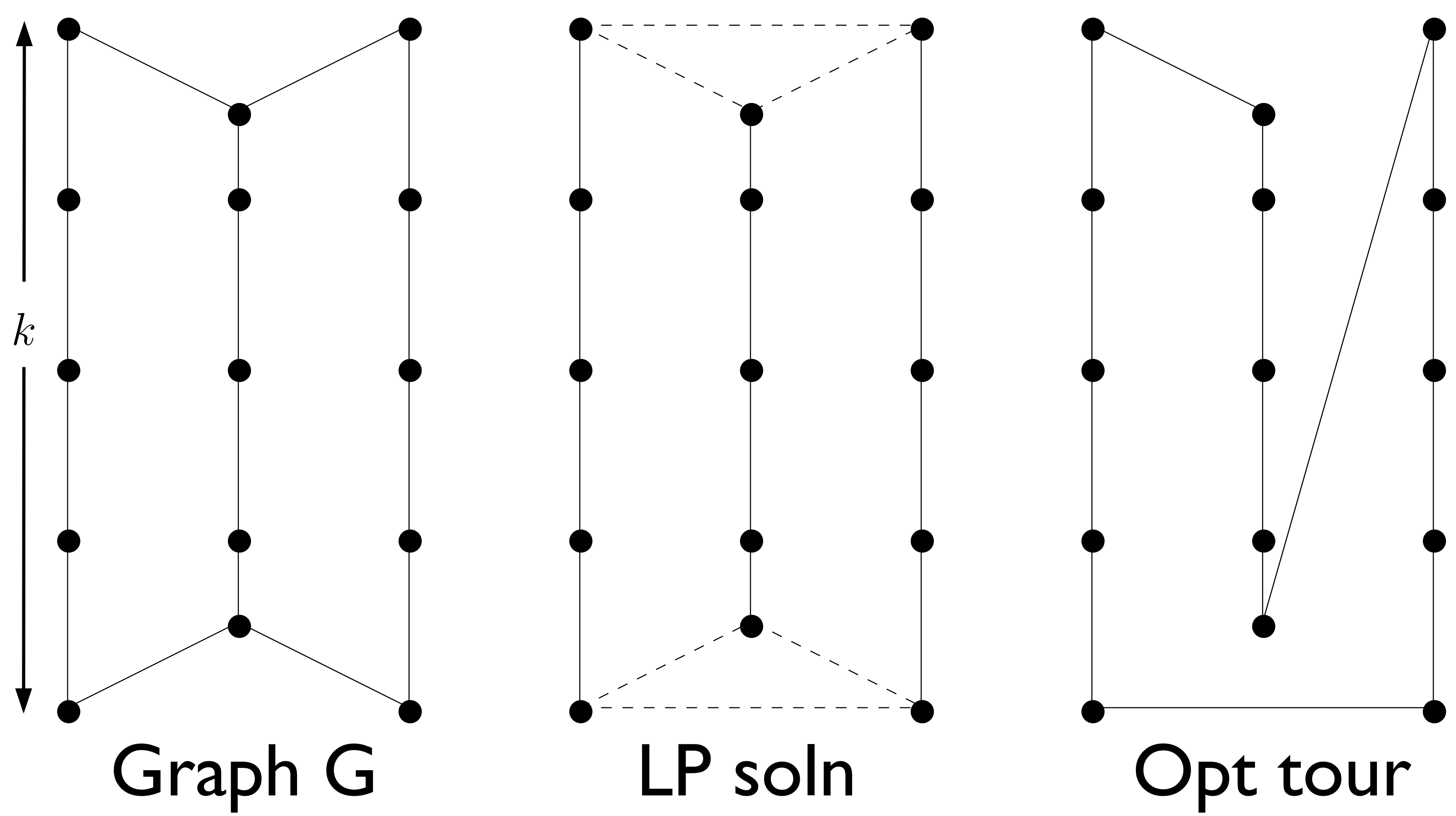
How strong is the Subtour LP bound?

- What about in theory?
- Define
  - SUBT(c) as the optimal value of the Subtour LP for costs c
  - OPT(c) as the length of the optimal tour for costs c
  - C_n is the set of all symmetric cost functions on *n* vertices that obey triangle inequality.
- Then the *integrality gap* of the Subtour LP is

$$\gamma \equiv \sup_n \gamma(n) \text{ where } \gamma(n) \equiv \sup_{c \in \mathcal{C}_n} \frac{OPT(c)}{SUBT(c)}$$

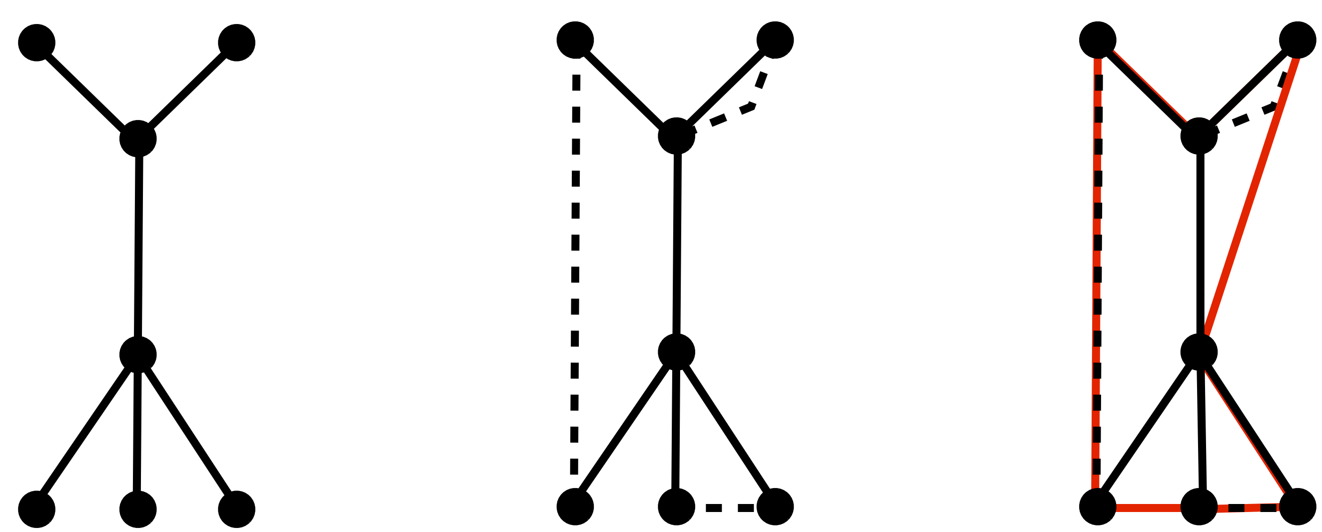
A lower bound

It's known that $\gamma \geq 4/3$, where *c(i,j)* comes from the shortest *i-j* path distance in a graph G (*graph TSP*).



Christofides' Algorithm

Christofides (1976) shows how to compute a tour in polynomial time of cost 3/2 optimal: compute a min-cost spanning tree, compute a matching on the odd-degree vertices, then "shortcut" a traversal of the resulting Eulerian graph.



≤ OPT(c) + ≤ 1/2 OPT(c) ≤ 3/2 OPT(c)

An upper bound

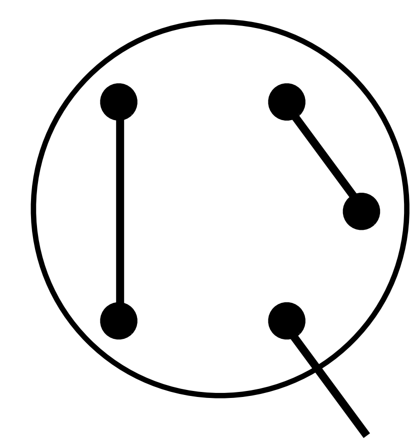
- Wolsey (1980) and Shmoys and W (1990) show that OPT(c) can be replaced with SUBT(c), so that Christofides gives a tour of cost $\leq$ 3/2 SUBT(c).
- Therefore,

$$OPT(c) \leq \frac{3}{2} SUBT(c) \quad \Rightarrow \quad \gamma \leq \frac{OPT(c)}{SUBT(c)} \leq \frac{3}{2}$$

Perfect Matching Polytope

Edmonds (1965) shows that the min-cost perfect matching can be found as the solution to the linear program:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} c(e) z(e) \\
\text{subject to} \quad & \sum_{e \in \delta(v)} z(e) = 1 \qquad \forall v \in V \\
& \sum_{e \in \delta(S)} z(e) \geq 1 \qquad \forall S \subset V, |S| \text{ odd}
\end{aligned}
$$



Matchings and the Subtour LP

Then MATCH(c) ≤ 1/2 SUBT(c) since z = 1/2 x is feasible for the matching LP.

$$\text{Minimize} \sum_{e \in E} c(e)x(e)$$

subject to

$$\sum_{e \in \delta(v)} x(e) = 2 \qquad \forall v \in V$$

$$\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subseteq V, |S| \geq 2$$

$$0 \leq x(e) \leq 1 \qquad \forall e \in E$$

$$\text{Minimize} \sum_{e \in E} c(e)z(e)$$

$$\text{subject to} \sum_{e \in \delta(v)} z(e) = 1 \qquad \forall v \in V$$

$$\sum_{e \in \delta(S)} z(e) \geq 1 \qquad \forall S \subset V, |S| \text{ odd}$$

Shmoys and W (1990) also show that SUBT(c) is nonincreasing as vertices are removed so that matching on odd-degree vertices is at most 1/2 SUBT(c).

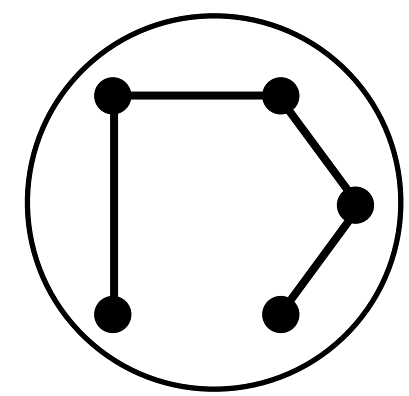
Spanning Tree Polytope

Similarly, Edmonds (1971) showed that the min-cost spanning tree can be found as the solution of the following LP:

$$\text{Minimize} \sum_{e \in E} c(e) z(e)$$

$$\text{subject to} \sum_{e \in E} z(e) = |V| - 1$$

$$\sum_{e \in E(S)} z(e) \leq |S| - 1 \qquad S \subset V$$



Spanning Trees and the Subtour LP

Then MST(c) ≤ ((n-1)/n) SUBT(c) since z = ((n-1)/n) x is feasible for the MST LP.

$$
\begin{aligned}
&\text{Minimize } \sum_{e\in E} c(e)x(e)\\
&\quad \text{subject to}\\
&\quad \sum_{e\in\delta(v)} x(e) = 2 \qquad \forall v \in V\\
&\quad \sum_{e\in E(S)} x(e) \le |S|-1 \qquad \forall S \subset V, |S| \ge 2\\
&\quad 0 \le x(e) \le 1 \qquad \forall e \in E
\end{aligned}
$$

$$
\begin{aligned}
&\text{Minimize } \sum_{e\in E} c(e)z(e)\\
&\text{subject to } \sum_{e\in E} z(e) = n-1\\
&\quad \sum_{e\in E(S)} z(e) \le |S|-1 \qquad S \subset V
\end{aligned}
$$

$$
\begin{aligned}
\sum_{e\in E} z(e) &= \frac{n-1}{n}\sum_{e\in E} x(e)\\
&= \frac{n-1}{n}\cdot\frac{1}{2}\sum_{v\in V}\sum_{e\in\delta(v)} x(e)\\
&= \frac{n-1}{n}\cdot\frac{1}{2}\cdot 2n\\
&= n-1
\end{aligned}
$$

Recent results

- Some recent progress on graph TSP (costs *c(i,j)* are the shortest *i-j* path distances in unweighted graph):
  - Boyd, Sitters, van der Ster, Stougie (2010); Aggarwal, Garg, Gupta (2011): Gap is at most 4/3 if graph is cubic.
  - Oveis Gharan, Saberi, Singh (2010): Gap is at most 3/2 - ε for a constant ε > 0.
  - Mömke, Svensson (2011): Gap is at most 1.461.
  - Mömke, Svensson (2011): Gap is 4/3 if graph is subcubic (degree at most 3).
  - Mucha (2011): Gap is at most 13/9 ≈ 1.44.
  - Sebő and Vygen (2012): Gap is at most 1.4.

Current state

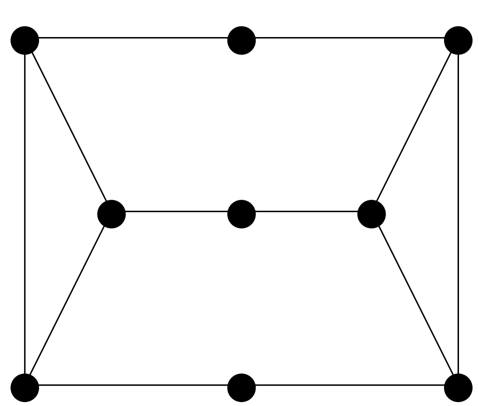
$$\frac{4}{3} \leq \gamma \leq \frac{3}{2}$$

- Conjecture (Goemans 1995, others): $\gamma = \frac{4}{3}$

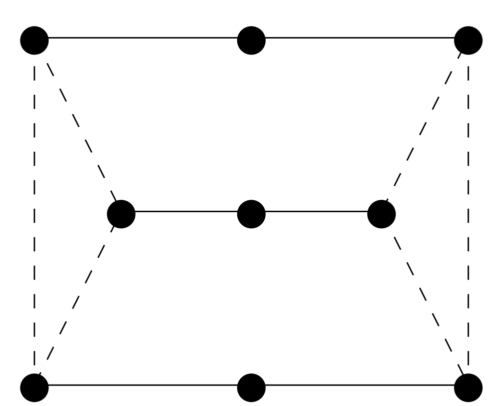
More ignorance

Let γ_{12} be the integrality gap for costs $c(i,j) \in \{1,2\}$. Then all we know is

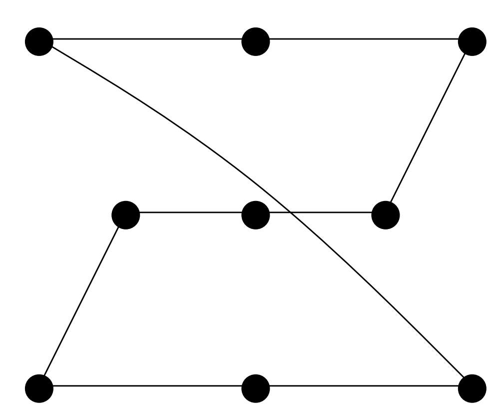
$$\frac{10}{9} \leq \gamma_{12} \leq \frac{3}{2}$$



cost 1 edges



LP soln



OPT

Still more ignorance

We don't even know the equivalent worst-case ratio between 2-matching costs 2M(c) and SUBT(c).

$$\mu \equiv \sup_n \mu(n) \text{ where } \mu(n) \equiv \sup_{c \in \mathcal{C}_n} \frac{2M(c)}{SUBT(c)}$$

Then all we know is that

$$\frac{10}{9} \leq \mu \leq \frac{4}{3} \text{ (Boyd, Carr 1999)}$$

Conjecture (Boyd, Carr 2011): $\mu = \frac{10}{9}$

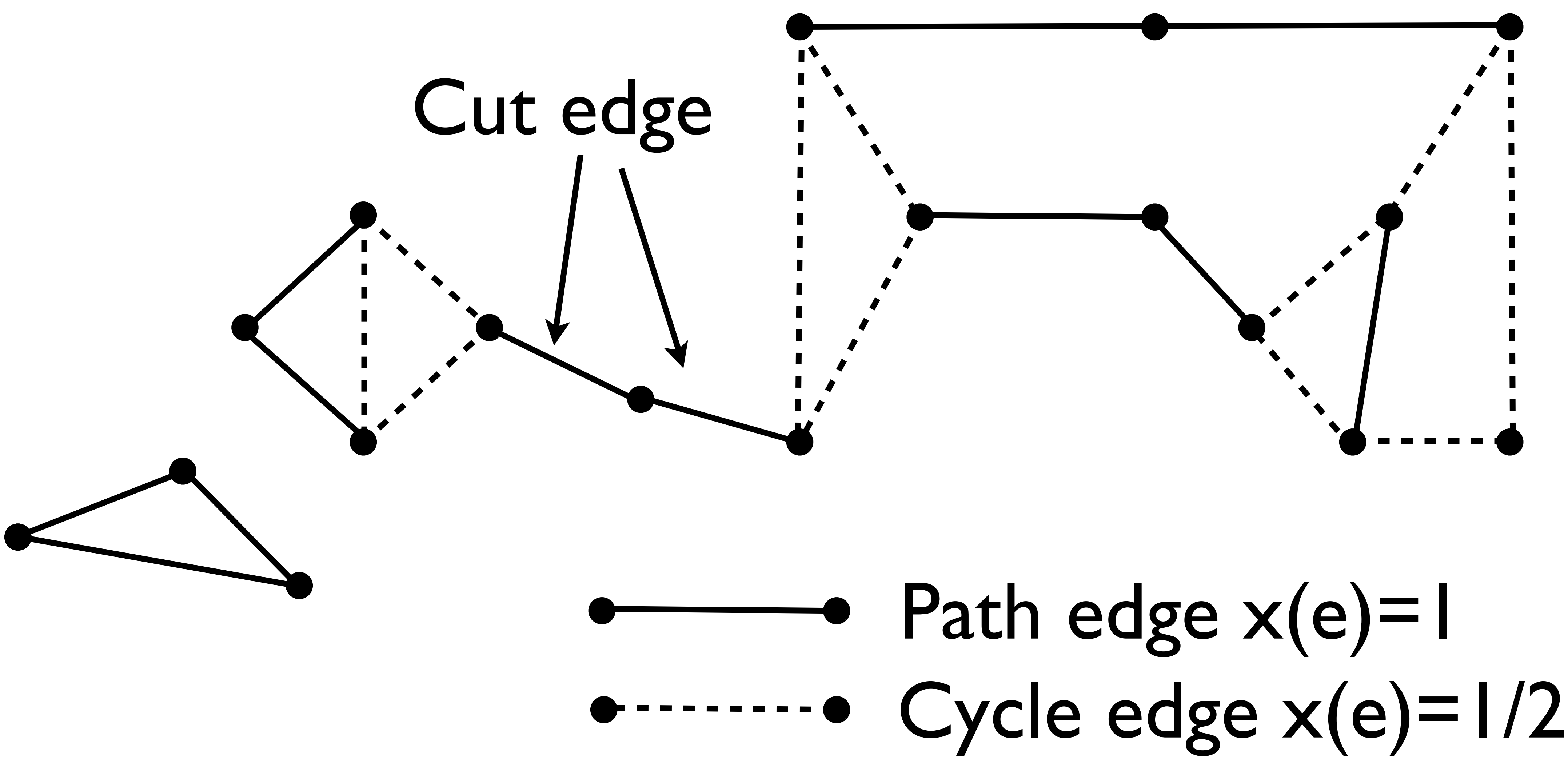
Our contributions

- We can prove the Boyd-Carr conjecture (with Schalekamp and van Zuylen)
- We can show $\gamma_{12} \leq 5/4$ (with Qian, Schalekamp, and van Zuylen).

Outline

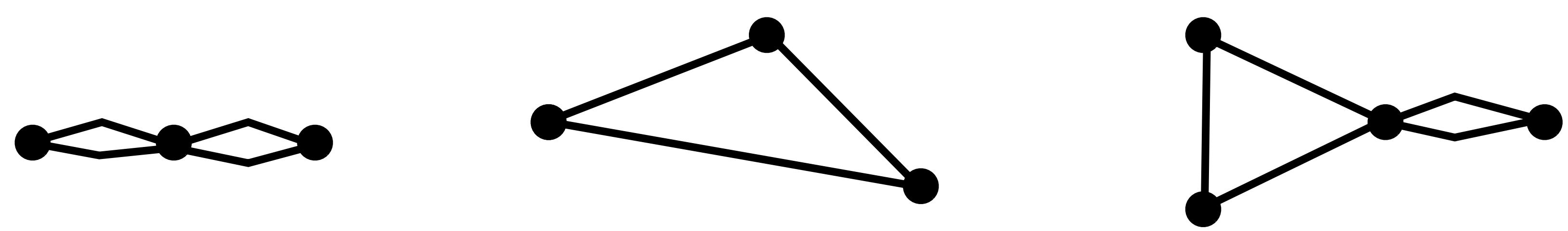
- A brief intro to the TSP
- A standard TSP linear program
  - Experimental analysis
  - Theoretical analysis: an outstanding open question
- A related question: the Boyd-Carr conjecture and its proof
  - $\mu \leq 4/3$ under a certain condition.
  - $\mu \leq 10/9$.
- Some conjectures and more experiments

Some terminology

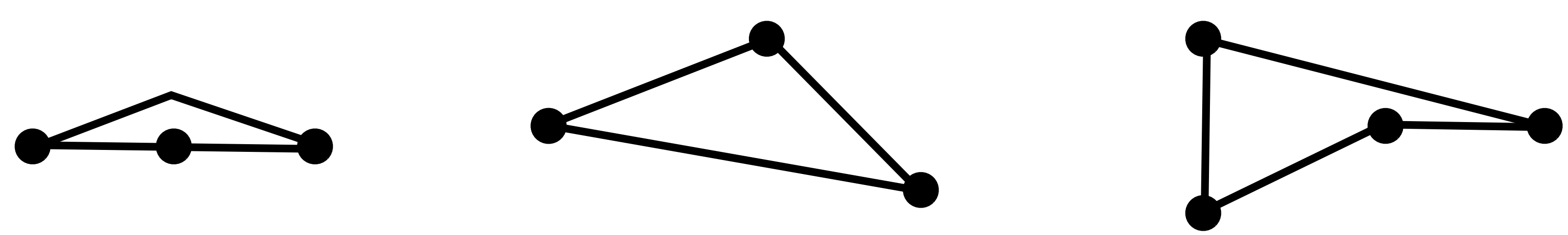


The strategy

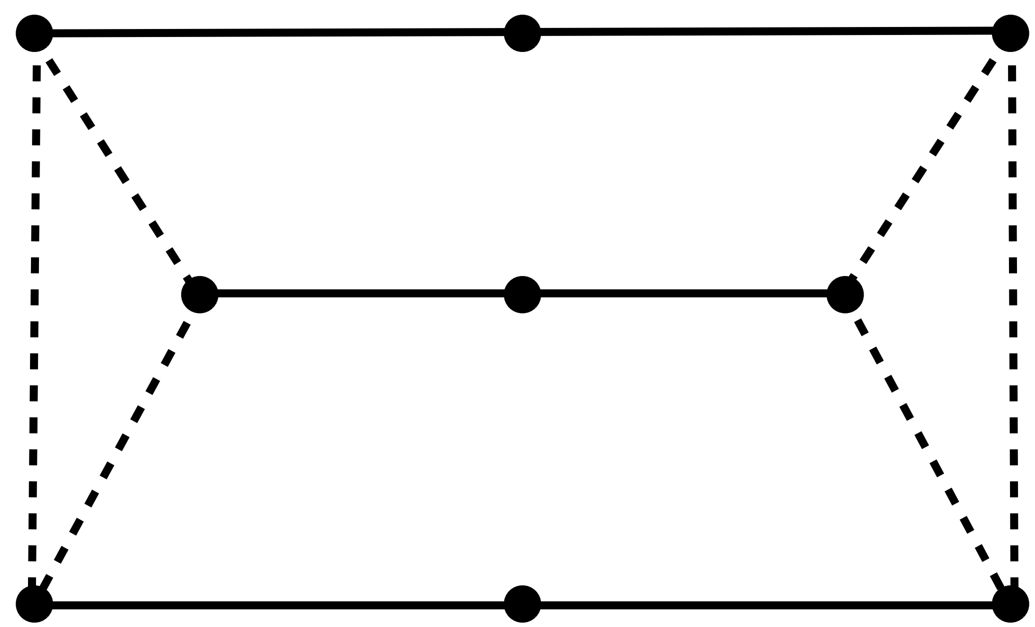
- Start with an optimal fractional 2-matching; this gives a lower bound on the Subtour LP.
- Add a low-cost set of edges to create a *graphical* 2-matching: each vertex has degree 2 or 4; each component has size at least 3; each edge has 0, 1, or 2 copies.



- "Shortcut" the graphical 2-matching to a 2-matching.

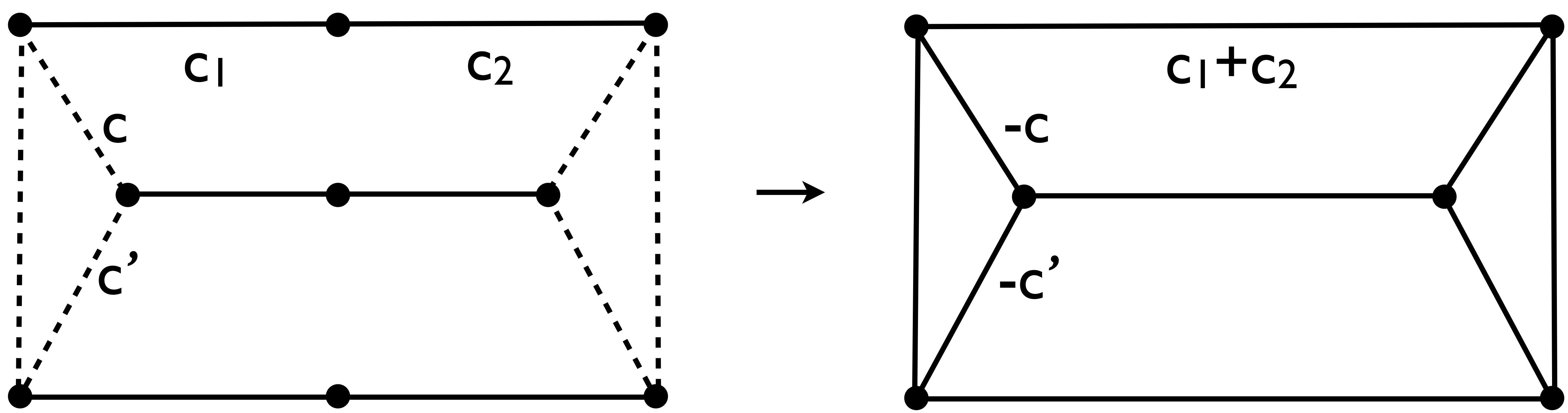


Consider fractional 2-matchings that have no cut edge; we show that we can get a graphical 2-matching with a 4/3 increase in cost.



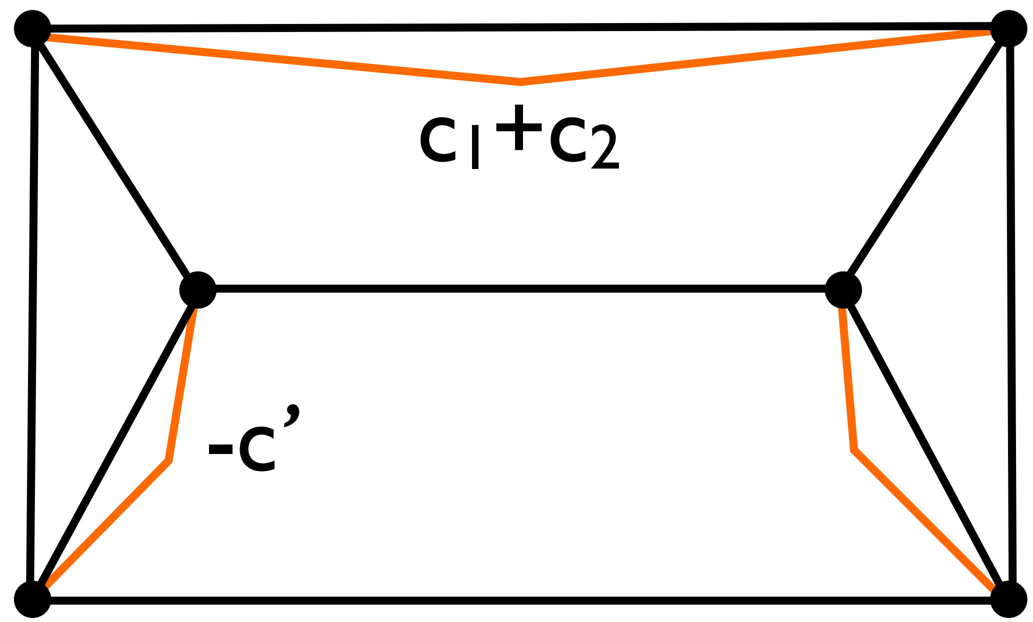
2M $\leq$ Graphical 2M $\leq$ 4/3 Fractional 2M $\leq$ 4/3 Subtour

Create new graph by replacing path edges with a single edge of cost equal to the path, cycle edges with negations of their cost.

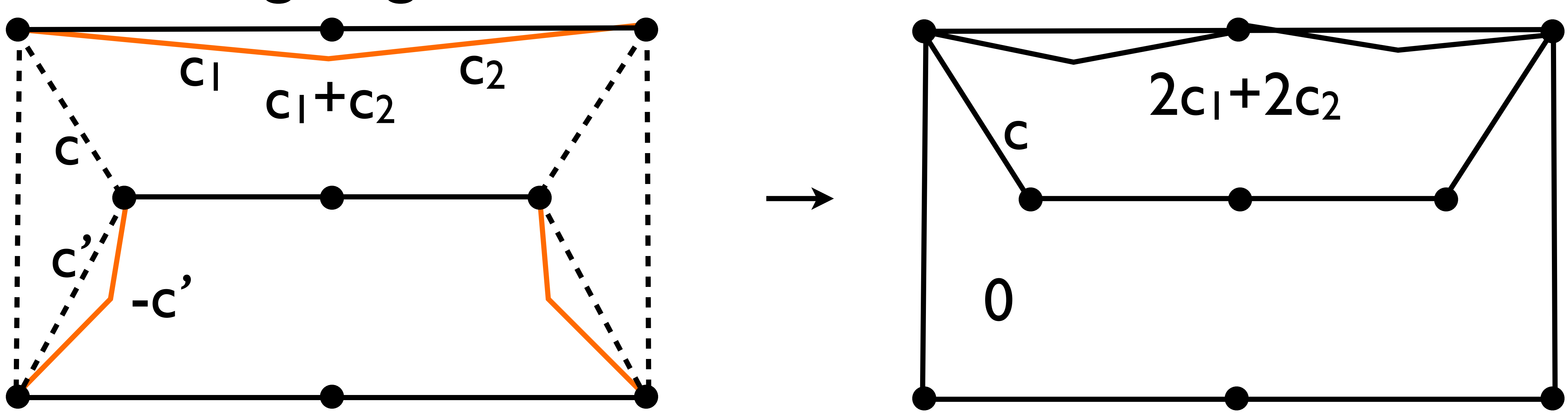


New graph is cubic and 2-edge connected.

Compute a min-cost perfect matching in new graph.

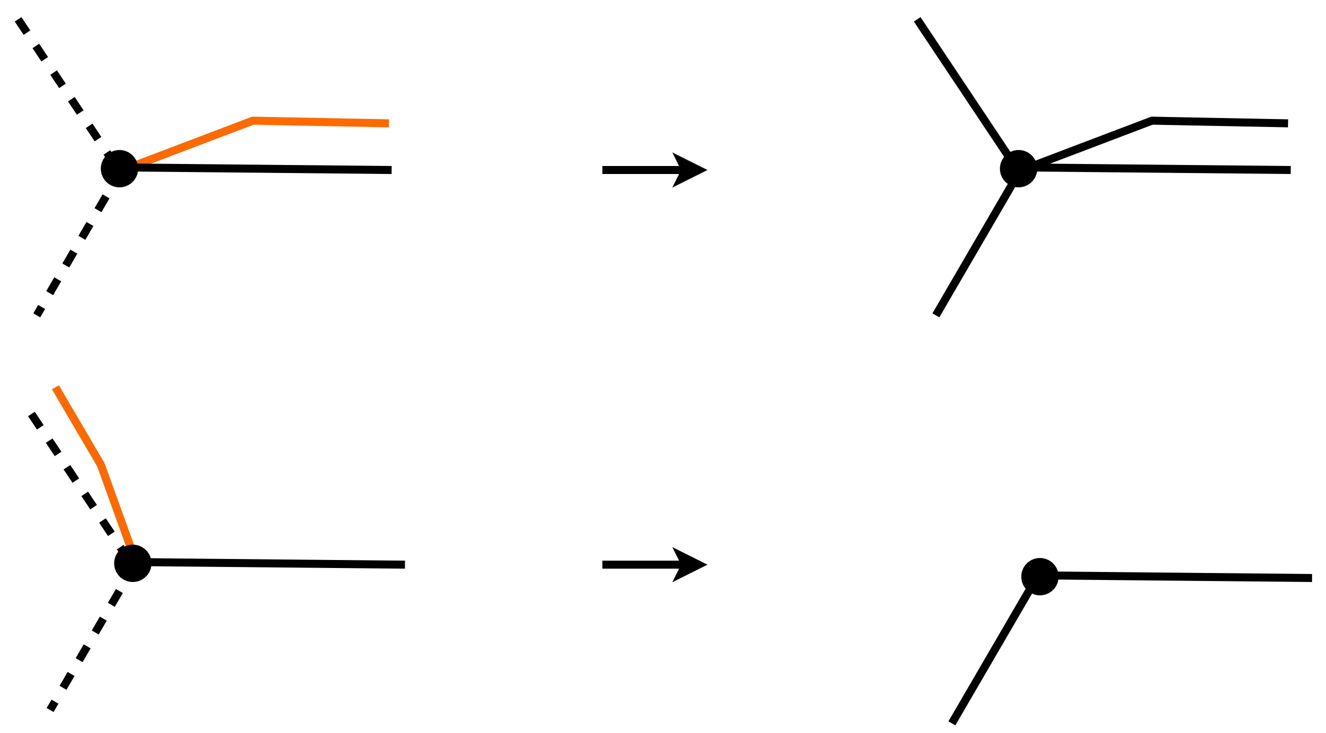


In the fractional 2-matching, double any path edge in matching, remove any cycle edge. Cost is paths + cycles + matching edges.



Why this works

For any given node on the cycle, either its associated path edge is in the matching or one of the two cycle edges.



Why this works

For any given node on the path, either its associated path edge is in the matching or not.

Bounding the cost

- P = total cost of all path edges
- C = total cost all cycle edges
- So fractional 2-matching costs P + C/2
- Claim: Perfect matching in the new graph costs at most 1/3 the cost of all its edges, so at most 1/3(P - C)

Bounding the cost

- Since the graphical 2-matching costs at most P + C + matching, it costs at most

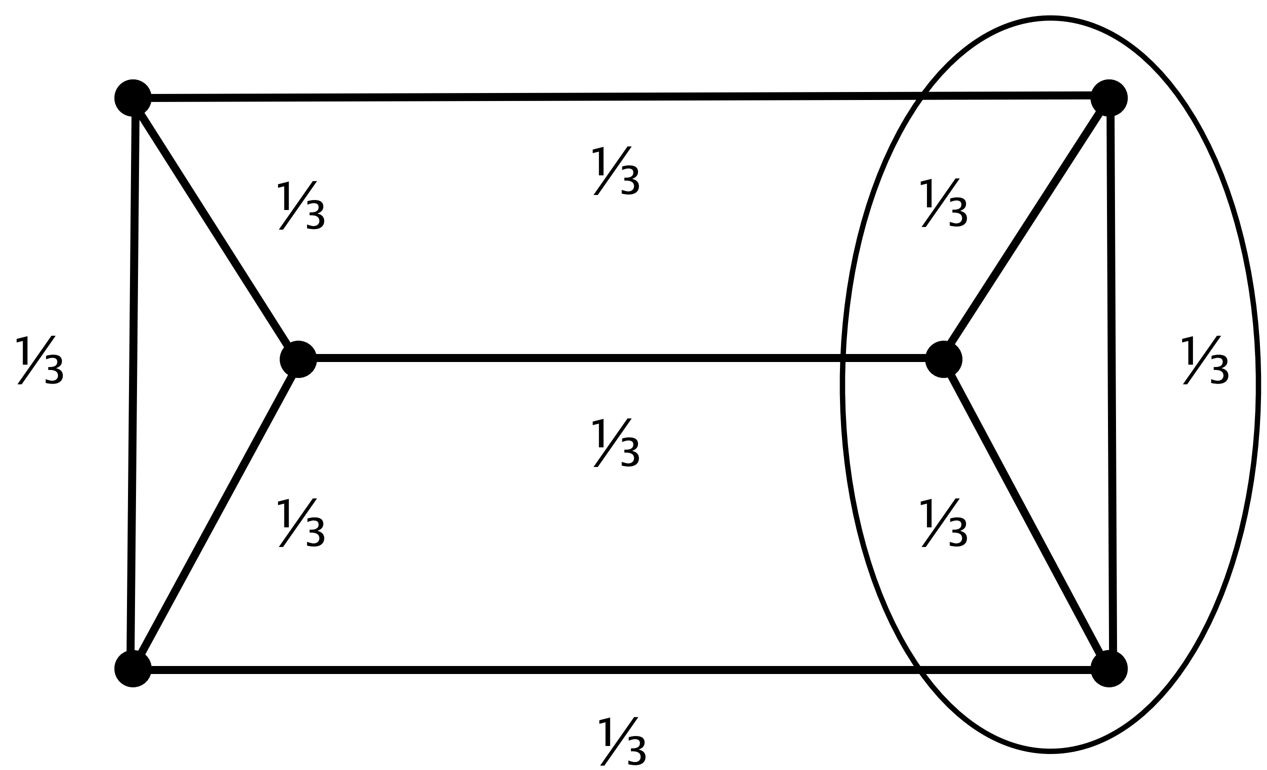
$$P + C + \frac{1}{3}(P - C) = \frac{4}{3}P + \frac{2}{3}C = \frac{4}{3}\left(P + \frac{1}{2}C\right)$$

2M $\leq$ Graphical 2M $\leq$ 4/3 Fractional 2M

$\leq$ 4/3 Subtour

Matching cost

- Naddef and Pulleyblank (1981): Any cubic, 2-edge-connected, weighted graph has a perfect matching of cost at most a third of the sum of the edge weights.
- Proof: Set z(e)=1/3 for all e∈E, then feasible for matching LP.

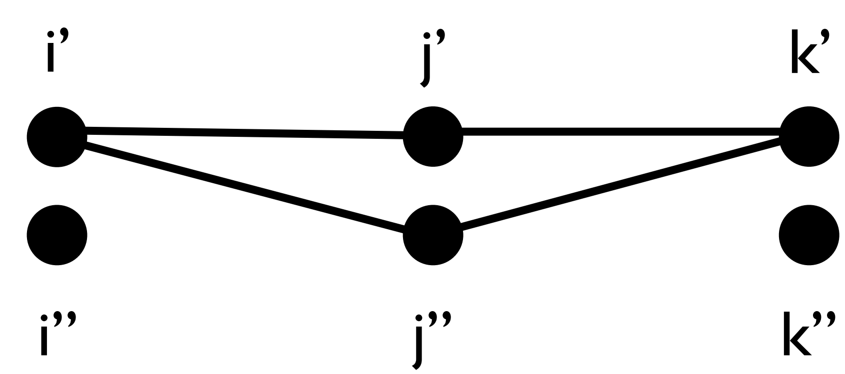


$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} c(e) z(e) & \\
\text{subject to} \quad & \sum_{e \in \delta(v)} z(e) = 1 & \forall v \in V \\
& \sum_{e \in \delta(S)} z(e) \geq 1 & \forall S \subset V, |S| \text{ odd}
\end{aligned}
$$

3|S| = 2|E(S)| + |δ(S)|, so for |S| odd, |δ(S)| odd.

Proving $\mu \leq 10/9$

- To prove stronger results, we give a polyhedral formulation for graphical 2-matchings.
- For all i∈V, create i’ and i’’
  - i’ *required*: must have degree 2
  - i’’ *optional*: may have degree 0 or 2
- For all (i,j)∈E, create edges (i’,j’), (i’,j’’), (i’’,j’)



The formulation

$$\sum_{e \in \delta(i')} y(e) = 2 \qquad \forall i'$$

$$\sum_{e \in \delta(i'')} y(e) \leq 2 \qquad \forall i''$$

$$\sum_{e \in \delta(S) - F} y(e) + |F| - \sum_{e \in F} y(e) \geq 1 \qquad \forall S \subseteq V, F \subseteq \delta(S), F \text{ matching}, |F| \text{ odd}$$

$$0 \leq y(e) \leq 1 \qquad \forall e \in E$$

Can show that the extreme points of this LP are graphical 2-matchings.

Proving μ ≤ 10/9

Given Subtour LP soln x, set

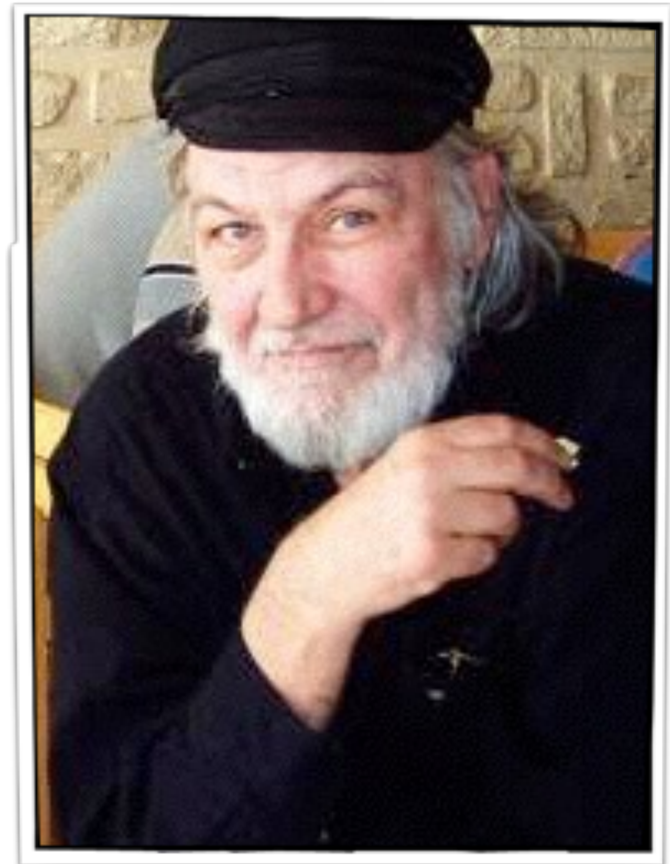
$$y(i',j') = \frac{8}{9}x(i,j)$$
$$y(i'',j') = \frac{1}{9}x(i,j)$$
$$y(i',j'') = \frac{1}{9}x(i,j)$$

$$\sum_{e\in\delta(i')} y(e) = 2 \qquad \forall i'$$
$$\sum_{e\in\delta(i'')} y(e) \le 2 \qquad \forall i''$$
$$\sum_{e\in\delta(S)-F} y(e) + |F| - \sum_{e\in F} y(e) \ge 1$$
$$\forall S \subseteq V, F \subseteq \delta(S), F \text{ matching}, |F| \text{ odd}$$
$$0 \le y(e) \le 1 \qquad \forall e \in E$$

$$\text{Minimize } \sum_{e\in E} c(e)x(e)$$

subject to

$$\sum_{e\in\delta(v)} x(e) = 2 \qquad \forall v \in V$$
$$\sum_{e\in\delta(S)} x(e) \ge 2 \qquad \forall S \subseteq V, |S| \ge 2$$
$$0 \le x(e) \le 1 \qquad \forall e \in E$$



Edmonds (1967)

traveling saleman problem [cf. 4]. I conjecture that there is no good algorithm for the traveling saleman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know.

A good algorithm is known for finding, in any graph

Some conjectures

- For the 1,2-TSP we conjecture that γ_{12} = 10/9. We show $\gamma_{12} \leq 5/4$.
- Computation shows the conjecture is true for n $\leq$ 12.

An observation

- We know
  - $\dfrac{2M(c)}{F2M(c)} \leq \dfrac{4}{3}$ (Boyd, Carr 1999)
  - $\dfrac{2M(c)}{F2M(c)} \leq \dfrac{10}{9} \quad \forall c \in \{1,2\}^n$ (this work)
- We conjecture $\gamma \leq 4/3$, $\gamma_{12} \leq 10/9$.
- Coincidence?

Another conjecture

- Conjecture: The worst case for the Subtour LP integrality gap (both γ and γ_{12}) occurs for solutions that are fractional 2-matchings.
- Note: we don't even know tight bounds on γ and γ_{12} in this case, though we can show $\gamma_{12} \leq 7/6$ in this case.

Best-of-Many Christofides'

A conjectured algorithm (Oveis Gharan, Saberi, Singh 2010; An, Kleinberg, Shmoys 2012):

- Solve Subtour LP for x.
- Since ((n-1)/n)x in spanning tree polytope, express ((n-1)/n)x as a convex combination of spanning trees.
- Sample a spanning tree from convex combination, run Christofides' algorithm on it.

Experimental Results

	Std	Max Entropy (Best)	Max Entropy (Ave)	Splitting Off (Best)	Splitting Off (Ave)
TSPLIB	9.45%	3.47%	6.29%	5.42%	6.34%
VLSI	9.87%	5.84%	7.52%	6.43%	7.42%
Graph	14.68%	0.51%	0.89%	1.34%	1.61%

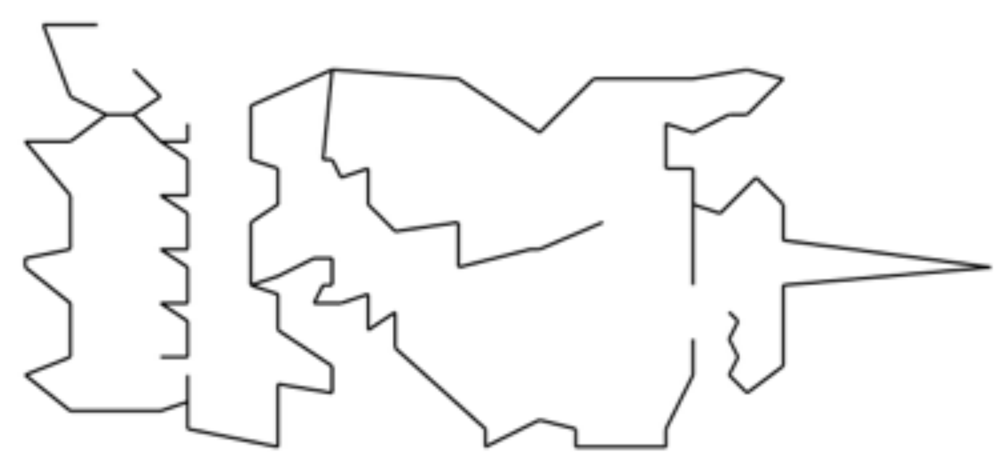
Percentages expressed with respect to cost of an optimal tour

(With Kyle Genova)

Why does this help?

Experimentally, almost all degrees of sampled spanning tree are two. The tree costs more, and matching edges are more expensive, but there are a lot fewer edges in the matching.

Standard Christofides'



Best-of-Many Christofides'

Experimental Results

	Tree		Matching		
	Std	Best of Many	Std	Max Entropy	Splitting Off
TSPLIB	87.85%	98.79%	31.06%	10.80%	10.63%
VLSI	90.57%	98.92%	29.46%	12.60%	12.49%
Graph	77.59%	98.78%	42.35%	3.16%	5.37%

Percentages expressed with respect to cost of an optimal tour

Analysis

- E[cost of tree] $\leq$ SUBT(c) by construction.
- Would need to show E[cost of matching] $\leq$ (1/2 - ε) SUBT(c) for some $\varepsilon > 0$.

“Theory is when we understand everything, but nothing works.

Practice is when everything works, but we don’t understand why.

At this station, theory and practice are united, so that nothing works and no one understands why.”

Thank you for your attention.